Ohta/Ono Lemma 6

Theorem 3 claims that many subspaces of the real line have the UMP. To prove this, the strategy is to create a subspace M and a metric d and prove that (M, d) has UMP. Both M and d are fairly complicated. Lemma 6 is the technical lemma — the lemma which verifies d truly is a topology-preserving metric and that (M, d) truly has the UMP.

As with any theorem/proof, "understanding" is a process. The goal is to know the statement AND know how the steps of the proof justify the statement AND know the conceptual framework of the proof. This is typically a multi-step process. When asked to "verify" in the questions below, aim for a higher level of understanding, but don't get bogged down. There are lots of facts!

M, the universal space

M is described on pages 219-220. It consists of the union of 3 sets: A, B, and C. Describe each of these sets – use words and figures to decipher the formal notation. Look at the conditions of Theorem 3 and determine how each component of X (the space Thm 3 wants to show has UMP) will fit inside M. Which part(s) of M will always be used, no matter the X? Begin pondering the question: Why are sets A, B, and C defined as they are?

d, the metric

d must produce a distance given any two points in M. Since there are distinct subsets of M, it isn't surprising there are multiple rules depending on where the two points lie. Look ahead to Fact (C.1) on page 223: two points in M are always less than 2 apart. This is a clue that d will be a bit unusual.

Definition of d, Part 1.

Clearly, the usual metric won't work on two points from set A. The function $h(x)$ is created and plays a role in defining d when the two points come from A. $h(x)$ is the first coordinate of the intersection point of the graphs of the functions $f(u) = 1 + e^u$ and $g(u) = x - u$. In symbols:

$h : A \to A$ where $1 + e^{h(x)} = x - h(x)$. Use your graphing calculator, CAS, or other means to explore $h(x)$. Answer the following questions. (See Maple output also.)

1. Is $h(x)$ well defined? That is, is there exactly one output for each input x in the domain of h?

2. Make a table of values and/or a sketch of $h(x)$ for, say, $-10 < x < -\ln 2$. What is a simple approximation for $h(x)$?

3. Verify the statement that $h(x) < x$ for each $x \in A$ using the definition of $h(x)$.

4. Verify (3.4): $0 < h(y) - h(x) < y - x$ if $x < y$. Hint: Use calculus.

5. Verify (3.11): $e^y - e^x \leq e^y(y - x)$ for each $x, y \in A$ with $x < y$. Hint: use Calc I again.

The distance between two points in A depends on whether the two points are "close" to each other or are "far apart." The h function is used to distinguish the two cases. If $x < y$ for $x, y \in A$, either $h(y) \leq x < y$ or $x < h(y) < y$.

1. Write down the rule for d in each of these two cases. Draw a visual aid to illustrate the distance d in each case. (The standard visual aid for the usual distance is a line segment the length of which is $|x - y|$.)

2. Verify Fact (C.1) for two points in A.

3. Verify Fact (A.1). Fix an x in A and choose $y \in A$ such that $y > x$. What happens to $d(x, y)$ as y increases? Then fix a z in A and let $x < z$. What happens to $d(x, z)$ as x increases?

   Thus, $d(x, \cdot)$ is monotone {increasing or decreasing} on $[x, -\ln 2]$ and $d(\cdot, z)$ is monotone {increasing or decreasing} on $(-\infty, z]$.

   Notation note: d is symmetric, so in the above statements, it doesn't matter the order of the dot and the variable. I chose to write the smaller quantity first. The named variable does not vary. The dot is a placeholder and takes on the values specified.

4. Verify Fact (E.2).

5. (optional) Verify Facts (C.2)-(C.4).

Definition of d, Part 2.

The rule for distance d between two points of $B \cup C$ is easy. Find it on page 220.

1. Verify Fact (C.1) for these points.

2. Verify Fact (C.6).

Definition of d, Part 3.

Now d needs to be defined for a point $x \in A$ and a point $y \in B$. The formula depends on the size of x and there are three cases. The formulas are given in (3.6)-(3.8). Note each has the form

$$1 + [\textit{something} + y] \cdot e^x$$

In a sense, there is a weighting on the e^x. The *something* depends both on x and on which interval B_i that y falls in. What is the largest e^x can be? Is the *something* positive or negative or 0? Is the magnitude of the *something* greater than or less than 1? Is $[\textit{something} + y]$ positive or negative?

Explore d where one point is in A and the other is in B. Suggestions:

1. Verify that the rules give the same distances between x and y when $x = -2, -3$.

2. (optional) Choose a specific x and y and calculate the distance between them.

3. Can you find a way to visualize d distances when one point is in A and the other is in B?

4. Fix a $z \in B$ and let x vary from $-\infty$ to $-\ln 2$. Do the distances increase or decrease as x increases? Compare your result to Fact (A.2).

5. Fix an $x \in A$ and let y vary within one B_i. What happens to $d(x, y)$? Compare your result to Fact (A.3).

6. Fix an $x \in A$ and let y jump from one B_i to another. Behavior is more complicated here! Note the sentence preceding Fact B.

Definition of d, Part 4.

Finally, d must be defined for $x < y$ where $x \in A$ and $y \in C$. This is similar to the previous case: the distance is again $1 + [\mathit{something} + y] \cdot e^x$, but the *something* is slightly different. Explore:

1. What is the sign and magnitude of $[\mathit{something} + y]$?
2. Verify that the rules give the same distances between x and y when $x = -2,\ -3$.
3. Fix a $z \in C$ and let x vary from $-\infty$ to $-\ln 2$. Do the distances increase or decrease as x increases? Compare your result to Fact (A.2).
4. Verify Fact (C.1) for $x \in A$ and $y \in B \cup C$.
5. Verify Fact (E.4).

More Facts.

1. Read the statements in Fact D. Verify one of them, say, (D.2).
2. Read the statements in Fact B. The function $\varphi(x) = d(x, y) - d(x, z)$ where $x \in A$ and $y, z \in B \cup C$ with $y \neq z$ is such that if $\varphi(x) = 0$, then x is the midpoint of y and z. Note that Fact (B.1) and (B.3) claim (in part) that $\varphi(x)$ is nonzero. Why is that interesting? Whether or not you verify any of these, look through the rest of the proof of Lemma 6 and find where these facts are used.
3. Verify any of the remaining facts you desire. Perhaps start with (C.5) and the rest of Fact E.

At this point, you should have a feel for d on M. Pause here and reflect on the purpose of Lemma 6 and how it will be used in proving Theorem 3. What remains to be done? Do you see the need for the plethora of Facts? How the Facts will be used? Jot down questions, comments, insights you have on M and d. Write a sentence or two describing the purpose of each of the five groups of facts.

A

B

C

D

E