Immerse Metric Space Homework

(Exercises 1-21)

1. In $\mathbb{R}^n$, define $d(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n|$. Show that d is a metric that induces the usual topology. Sketch the basis elements when $n = 2$.

   Solution: Steps (a) through (d) show that $d(x,y)$ is a metric.

   (a) We want to show that the distance between any two points is greater than or equal to zero. We know that each $|x_i - y_i|$ is greater than or equal to 0 by the definition of absolute value. So the sum $d(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n|$ is greater than or equal to 0.

   (b) Next we want to show that $d(x,y) = 0 \iff x = y$.

   $\Rightarrow$ Suppose that $d(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n| = 0$; then each $|x_i - y_i| = 0$ for all $i = 1 \ldots n$. So $x_i - y_i = 0$ and $x_i = y_i$ for all $i = 1 \ldots n$.

   $\Leftarrow$ Suppose that $y = x$. Then for $d(x,y) = |x_1 - y_1| + \ldots + |x_n - y_n|$ we can plug x_i in for y_i, and we get $d(x,y) = |x_1 - x_1| + \ldots + |x_n - x_n| = 0$.

   (c) Now we want to show that $d(x,y) = d(y,x)$.

   Assume $d(x,y) \neq d(y,x)$. Let $m_i = x_i - y_i$ for all $i = 1, \ldots, n$. Then $|x_1 - y_1| + \ldots + |x_n + y_n| \neq |y_1 - x_1| + \ldots + |y_n - x_n|$, implying that $|m_1| + \ldots + |m_n| \neq |-m_1| + \ldots + |-m_n|$. We have reached a contradiction, since by definition $|m| = |-m|$. Therefore, $d(x,y) = d(y,x)$.

   (d) Finally we want to show that the triangle inequality holds for d. We know that $|x_i - y_i| = |x_i - z_i + z_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$. Since this holds for all $i = 1, \ldots, n$ we know that $d(x,y) = \sum_{i=1}^{n} |x_i - y_i| \leq \sum_{i=1}^{n} |x_i - z_i| + |z_i - y_i| = d(x,z) + d(z,y)$. Thus the triangle inequality holds for d.

   Now to show that $d(x,y)$ induces the usual topology, let δ be the usual metric, let $B_\delta(x,\varepsilon)$ be given, and let y be in the open ball centered at x of radius $\frac{\varepsilon}{\sqrt{n}}$, that is $y \in B_d(x, \frac{\varepsilon}{\sqrt{n}})$. Then we know that $|x_1 - y_1| + \ldots + |x_n + y_n| < \frac{\varepsilon}{\sqrt{n}}$, which implies that each individual $|x_i - y_i| < \frac{\varepsilon}{\sqrt{n}}$ for all $i = 1, \ldots, n$. Therefore:

   $$\sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < \sqrt{\sum_{i=1}^{n} \frac{\varepsilon^2}{n}} = \varepsilon$$

   Hence,$y \in B_\delta(x,\varepsilon)$, which means that y is in an open ball in the topology induced by $d(x,y)$, implies that y is in an open ball in usual topology. Now if we let $B_d(x,\varepsilon)$ be given, and let $y \in B_\delta(x, (\frac{\varepsilon}{n})^2)$, which implies that $(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2 < (\frac{\varepsilon}{n})^2$. Then we know that each $|x_i - y_i|^2 < (\frac{\varepsilon}{n})^2$ for all $i = 1, \ldots, n$, implying that $|x_i - y_i| < \frac{\varepsilon}{n}$ for all $i = 1, \ldots, n$. Therefore:

   $$\sum_{i=1}^{n} |x_i - y_i| < \varepsilon$$

Thus, $y \in B_d(x, \varepsilon)$. Thus, if y is in an open ball in the usual topology, then y is in an open ball in the topology induced by $d(x, y)$.

2. In $\mathbb{R}^n$, define $d(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p}$. Assume that d is a metric, and show that d induces the usual topology.

We want to show that $\tau_d \sim \tau_{Euc}$ where τ_d is the topology induced by the metric d and τ_{Euc} is the topology induced by the usual Euclidean metric. We know by exercise (1) that $\tau_{Euc} \sim \tau_{abs}$ where τ_{abs} is the topology induced by the absolute value metric as defined in exercise (1). So we will prove that $\tau_d \sim \tau_{abs}$. Let $y \in B_d(x, \epsilon)$. Then $(\sum_{i=1}^{n} |x_i - y_i|^p)^{1/p} < \epsilon$. So $\sum_{i=1}^{n} |x_i - y_i|^p < \epsilon^p$. So $|x_i - y_i|^p < \epsilon^p \; \forall i = 1, ..., n$. Then $|x_i - y_i| < \epsilon \; \forall i = 1, ..., n$.

So $\sum_{i=1}^{n} |x_i - y_i| < n\epsilon$. So $y \in B_{abs}(x, n\epsilon)$. Now let $z \in B_{abs}(x, \epsilon)$.

So $\sum_{i=1}^{n} |x_i - z_i| < \epsilon$. Then $(\sum_{i=1}^{n} |x_i - z_i|)^p < \epsilon^p$.

We know $\sum_{i=1}^{n} |x_i - z_i|^p \leq (\sum_{i=1}^{n} |x_i - z_i|)^p$. So $(\sum_{i=1}^{n} |x_i - z_i|^p)^{1/p} \leq \sum_{i=1}^{n} |x_i - z_i| < \epsilon$. So $z \in B_d(x, \epsilon)$. Hence $\tau_d \sim \tau_{abs} \sim \tau_{Euc}$ as desired.

3. Show that the topology induced by a metric d is the coarsest topology relative to which the metric is continuous

Proof. (First, show If d is continuous with metric τ, then $\tau_d \subseteq \tau$.)

Assume that d is continuous with respect to τ. Let B be an open set in τ_d such that $B = B(x_0, \epsilon)$ for some $x_0 \in X$.
Show B is open in (X, τ).
$d : X \times X \rightarrow \Re^+$ is continuous wrt τ.
Define $d' : X \rightarrow \Re^+$ to be $d'(x) = d(x, x_0)$.
So $d' = d_{|(X \times x_0)}$, so d' is continuous wrt to τ.
Note that $[0, \epsilon)$ is open in $\Re^+$. Then $B = B(x_0, \epsilon) = d'^{-1}([0, \epsilon]) \subseteq (X, \tau)$.

(Next show τ_d makes d continuous.)
Let $(a, b) \in \Re^+$.
Show $d^{-1}((a, b))$ is open in τ_d.
Let $U = d^{-1}((a, b))$.

Let $\overline{x} \in d^{-1}((a,b))$, so $\overline{x}$ is a point (x,y).
$d(x,y) \in (a,b)$.
Let $\epsilon = \frac{1}{2}min\{|a-d(x,y)|, |b-d(x,y)|\}$.
Let $B_1 = B(x,\epsilon) \subseteq X$and $B_2 = B(y,\epsilon) \subseteq X$.
Show $B_1 \times B_2 \subset U$.
Let $z \in B_1 \times B_2$, where $z = (x',y')$.
Show $a < d(x',y') < b$.
$d(x',y') \leq d(x',x) + d(x,y) + d(y,y')$.
Since $d(x',x) < \epsilon \leq \frac{1}{2}(|b-d(x,y)|)$
and $d(y,y') < \epsilon \leq \frac{1}{2}(|b-d(x,y)|)$, $d(x',y') < b$.
$d(x',y') \geq d(x,y) - (d(x',x) + d(y',y)) > a$, by the inverse triangle-inequality.
Thus $z \in U$, so $U = d^{-1}((a,b))$ is open.
Therefore, d is continuous on τ_d.

□

4. Let d be a metric and let $d'(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$. Prove that d' is a bounded metric.

Claim: If $\alpha \leq \beta$ and $\alpha, \beta > 0$, then $\dfrac{\alpha}{1+\alpha} \leq \dfrac{\beta}{1+\beta}$.

Proof. (Proof of claim) Let $\alpha \leq \beta$ and $\alpha, \beta > 0$. Adding $\alpha\beta$ to both sides we obtain, $\alpha + \alpha\beta \leq \beta + \alpha\beta$. Now factoring out both sides, we get $\alpha(1+\beta) \leq \beta(1+\alpha)$. Finally, since $\alpha, \beta > 0$, we can divide both sides and we are left with, $\dfrac{\alpha}{1+\alpha} \leq \dfrac{\beta}{1+\beta}$.

□

Proof. (Proof that d' is a metric) Let $p,q,r \in X$.
1. $d'(p,q) = \dfrac{d(p,q)}{1+d(p,q)} \geq 0$ since $d(p,q) \geq 0$ and $1 + d(p,q) > 0$. Therefore, $d'(p,q) \geq 0$.
2. We know that $d'(p,q) = \dfrac{d(p,q)}{1+d(p,q)} = 0$ if and only if $d(p,q) = 0$. Since d is a metric, $d(p,q) = 0$ if and only if $p = q$. Therefore, $d'(p,q) = 0$ if and only if $p = q$.
3. Since d is a metric, $d(p,q) = d(q,p)$. So, $d'(p,q) = \dfrac{d(p,q)}{1+d(p,q)} = \dfrac{d(q,p)}{1+d(q,p)} = d'(q,p)$
Therefore, $d'(p,q) = d'(q,p)$.
4. Since d is a metric, $d(p,q) \leq d(p,r) + d(r,q)$. So by the above Lemma,
$$d'(p,q) = \frac{d(p,q)}{1+d(p,q)} \leq \frac{d(p,r)+d(r,q)}{1+d(p,r)+d(r,q)} = \frac{d(p,r)}{1+d(p,r)+d(r,q)} + \frac{d(r,q)}{1+d(p,r)+d(r,q)}.$$
Since d is a metric, $d(p,r), d(r,q) \geq 0$. Therefore $\dfrac{d(p,r)}{1+d(p,r)+d(r,q)} \leq d'(p,r)$ and $\dfrac{d(r,q)}{1+d(p,r)+d(r,q)} \leq d'(r,q)$.

Thus, $d'(p,q) \leq \dfrac{d(p,r)}{1+d(p,r)+d(r,q)} + \dfrac{d(r,q)}{1+d(p,r)+d(r,q)} \leq d'(p,r)+d'(r,q)$. Therefore, $d'(p,q) \leq d'(p,r)+d'(r,q)$.
Thus d' is a metric. ☐

Proof. (Proof that d' is bounded) Let $p,q \in X$. Let $M = 1$. Since d is a metric, $d(p,q) \geq 0$. Therefore, $d(p,q) \geq 0$, $1+d(p,q) > 0$ and $d(p,q) < 1+d(p,q)$. Hence, $d'(p,q) = \dfrac{d(p,q)}{1+d(p,q)} < 1 = M$. Therefore, d' is bounded by M. ☐

5. Let d be a metric. Show that $\bar{d}(x,y) = \min\{d(x,y),1\}$ induces the same topology as d.

Proof. Let τ_d and $\tau_{\bar{d}}$ be the topologies induced by d and $\bar{d}$ respectively. To show that $\bar{d}$ induces the same topology as d, we will show containment of open sets in τ_d in $\tau_{\bar{d}}$, and vice-versa.

Let $U \in \tau_d$ and fix $\rho \in U$. Then there exists $r > 0$ such that $B_d(\rho, r) \subseteq U$, by definition of an open set. Let $r' = min\{r, 1\}$. Then $B_{\bar{d}}(\rho, r') = B_d(\rho, r') \subseteq B_d(\rho, r) \subseteq U$. So $U \in \tau_{\bar{d}}$.

Now let $U \in \tau_{\bar{d}}$. Fix $\rho \in U$. Then there exists $r' > 0$ such that $B_{\bar{d}}(\rho, r') \subseteq U$. We may suppose that $r' \leq 1$ (else $B_{\bar{d}}(\rho, r') = X \subseteq U$, which implies that $U = X$, open in τ_d). So $B_d(\rho, r') = B_{\bar{d}}(\rho, r') \subseteq U$ and $U \in \tau_d$.

We have shown double containment of open sets in τ_d and $\tau_{\bar{d}}$, therefore we have that $\bar{d}(x,y) = \min\{d(x,y),1\}$ induces the same topology as d. ☐

6. For x and y in R^n, let $x \cdot y = \sum x_i y_i$ and $\|x\| = \sqrt{x \cdot x}$. Show that the Euclidean metric d on R^n is a metric by completing the following:.

   (a) Show that $(x \cdot y) + (x \cdot z) = x \cdot (y+z)$.

   PROOF.

$$\begin{aligned}(x \cdot y) + (x \cdot z) &= \sum x_i y_i + \sum x_i z_i \\ &= \sum (x_i y_i + x_i z_i) \\ &= \sum [x_i(y_i + z_i)] \\ &= x \cdot (y+z)\end{aligned}$$

   (b) We need to show $|\overrightarrow{x} \cdot \overrightarrow{y}| \leq \|x\|\|y\| \forall \overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$

Proof. We note that

$$\begin{aligned}
0 &\leq \|\overrightarrow{x} - t\overrightarrow{y}\|^2 \\
&= |\overrightarrow{x} - t\overrightarrow{y}| \cdot |\overrightarrow{x} - t\overrightarrow{y}| \\
&= \overrightarrow{x} \cdot |\overrightarrow{x} - t\overrightarrow{y}| - t\overrightarrow{y}|\overrightarrow{x} - t\overrightarrow{y}| \\
&= \overrightarrow{x} \cdot \overrightarrow{x} - t\overrightarrow{x} \cdot \overrightarrow{y} - t\overrightarrow{y} \cdot \overrightarrow{y} + t^2\overrightarrow{y}\,\overrightarrow{y} \\
&= \|\overrightarrow{x}\|^2 - 2t\overrightarrow{x} \cdot \overrightarrow{y} + t^2\|y\|^2
\end{aligned}$$

If $\overrightarrow{y} = 0$, we have shown the inequality. Assume that $y \neq 0$. Let $t = \frac{\overrightarrow{x} \cdot \overrightarrow{y}}{\|\overrightarrow{y}\|^2}$. Then we have from above

$$\begin{aligned}
0 &\leq \|x\|^2 - 2\frac{\overrightarrow{x} \cdot \overrightarrow{y}}{\|\overrightarrow{y}\|^2}\overrightarrow{x} \cdot \overrightarrow{y} + \left(\frac{\overrightarrow{x} \cdot \overrightarrow{y}}{\|\overrightarrow{y}\|^2}\right)^2 \|y\|^2 \\
&\leq \|x\|^2 - 2\frac{(\overrightarrow{x} \cdot \overrightarrow{y})^2}{\|\overrightarrow{y}\|^2} + \frac{(\overrightarrow{x} \cdot \overrightarrow{y})^2}{\|\overrightarrow{y}\|^2} \\
&= \|x\|^2 - \frac{(\overrightarrow{x} \cdot \overrightarrow{y})^2}{\|\overrightarrow{y}\|^2}
\end{aligned}$$

Then, we have

$$(\overrightarrow{x} \cdot \overrightarrow{y})^2 \leq \|\overrightarrow{x}\|^2\|\overrightarrow{y}\|^2.$$

Thus taking the square root of both sides we have,

$$(\overrightarrow{x} \cdot \overrightarrow{y}) \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$$

□

(c) Show that $\|x + y\| \leq \|x\| + \|y\|$.

PROOF.

$$\begin{aligned}
\|x + y\| &= \sqrt{(x + y) \cdot (x + y)} \\
\|x + y\|^2 &= \sqrt{(x + y) \cdot (x + y)}^2 \\
&= \sum_{i=1}^{n} (x_i + y_i)^2 \\
&= \sum_{i=1}^{n} (x_i^2 + 2x_iy_i + y_i^2) \\
&= \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} 2x_iy_i + \sum_{i=1}^{n} y_i^2 \\
&= \|x\|^2 + 2\sum_{i=1}^{n} x_iy_i + \|y\|^2 \\
&\leq \|x\|^2 + 2|x \cdot y| + \|y\|^2 \\
&\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{by part (c)} \\
&= (\|x\| + \|y\|)^2
\end{aligned}$$

Taking the squareroot of both sides we achieve the desired result.

(d) Verify that $d(x,y) = \|x-y\|$ is a metric.

PROOF.
To verify that d is a metric it must satisfy the following four properties.

i. $d(x,y) \geq 0$

Proof.

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \\ &\geq 0 \end{aligned}$$

ii. $d(x,y) = 0 \Leftrightarrow x = y$

Proof.

$$\begin{aligned} d(x,y) &= 0 \\ \Leftrightarrow \|x-y\| &= 0 \\ \Leftrightarrow \sqrt{(x-y)\cdot(x-y)} &= 0 \\ \Leftrightarrow \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} &= 0 \\ \Leftrightarrow x_i &= y_i \;\; \forall i \end{aligned}$$

iii. $d(x,y) = d(y,x)$

Proof.

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \\ &= \sqrt{\sum_{i=1}^{n}(y_i-x_i)^2} \\ &= \|y-x\| \\ &= d(y,x) \end{aligned}$$

iv. $d(x, y) \leq d(x, r) + d(r, y)$

Proof.

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \|(x-r)+(r-y)\| \\ &\leq \|x-r\| + \|r-y\| \text{ (by part (c))} \\ &= d(x,r) + d(r,y) \end{aligned}$$

By verifying these properties we have proven that the Euclidean metric d on R^n is a metric.

7. Prove the continuity of the algebraic operations of the real line.

Lemma: Let $f, g : X \to Y$ be continuous functions. Then the function $h : X \times X \to Y \times Y$ defined by $h(x, y) = (f(x), g(y))$ is continuous.

From the lecture notes theorem part (i) it is sufficient to prove that the function $F : X \times X \to Y$ defined by $F(x, y) = f(x)$ is continuous (and similarly the function $G : X \times X \to Y$ defined by $G(x, y) = g(y)$ is continuous). Let U be an open set in Y. Then $f^{-1}(U)$ is open in X. But $F^{-1}(U) = f^{-1}(U) \times X$. Since $f^{-1}(U)$ and X are both open in X, $f^{-1}(U) \times X$ is open in $X \times X$, so F is continuous.

Proof.

(a) *Addition:* Let (a, b) be a basis element of $\mathbb{R}$. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the addition function. Then $f^{-1}[(a, b)] = \{(x, y) \in \mathbb{R} \times \mathbb{R} \,|\, x + y \in (a, b)\}$.
Fix any $(c, d) \in f^{-1}[(a, b)]$ and let $\epsilon = \min\{(c+d-a), (b-(c+d))\}$. We want to show that there is an open set containing (c, d) that is contained within $f^{-1}[(a, b)]$.
Let d_1^2 be the taxicab metric, which we know induces the usual topology on $\mathbb{R}\times\mathbb{R}$.
Then $B_{d_1^2}((c, d), \epsilon/2)$ is an open set that that contains the point (c, d).
Let $(w, z) \in B_{d_1^2}((c, d), \epsilon/2)$. Then $d_1^2((w, z), (c, d)) = |w-c|+|z-d| < \epsilon/2$, which implies that both $|w - c|$ and $|z - d|$ are less than $\epsilon/2$. So $w + z < \epsilon + c + d \leq b - (c + d) + c + d = b$ and $w + z > c + d - \epsilon \geq c + d - (c + d - a) = a$. So $w + z \in (a, b)$, which implies that $B_{d_1^2}((c, d), \epsilon/2) \subseteq f^{-1}[(a, b)]$. Therefore $f^{-1}[(a, b)]$ is open, which implies that f is continuous.

(b) *Subtraction:* It is enough to show that the additive inverse map $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = -x$ is continuous. Then the subtraction map $s : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be written as $s(x, y) = f(i(x), g(y))$ where $i(x)$ is the identity map (continuous by lecture theorem part (b)) and $f(x, y)$ is the addition map which we have shown to be continuous. Then by our lemma s is the composition of continuous functions is itself continuous.
Fix any $x \in \mathbb{R}$ and any $\epsilon > 0$. If $|x - y| < \epsilon$, then $|g(x) - g(y)| = |-x - (-y)| = |(-1)(x-y)| = |x-y| < \epsilon$. So by our $\epsilon-\delta$ definition of continuity g is continuous.

(c) *Multiplication* Fix any $\epsilon > 0$ and any point $(x,y) \in \mathbb{R} \times \mathbb{R}$. Then set $\delta = \frac{-m+\sqrt{m^2+4\epsilon}}{2} > 0$, where $m = \max\{|x|,|y|\}$. Consider any $(x_n, y_n) \in B_{d_1^2}((x,y),\delta)$. Then

$$\begin{aligned} |x - x_n| + |y - y_n| &< \delta \\ |x-x_n|^2 + 2|x-x_n||y-y_n| + |y-y_n|^2 &< \delta^2 \\ |x-x_n||y-y_n| &< \delta^2. \end{aligned}$$

From our definition of m we know

$$|y||x-x_n| + |x||y-y_n| \leq m(|x-x_n| + |y-y_n|) < m\delta.$$

We note that

$$\delta^2 + m\delta = \left(\frac{m^2}{4} - \frac{m\sqrt{m^2+4\epsilon}}{2} + \frac{m^2+4\epsilon}{4}\right) + \left(\frac{-m^2}{2} + \frac{m\sqrt{m^2+4\epsilon}}{2}\right) = \epsilon.$$

Combining these results and using the triangle inequality we have

$$\begin{aligned} |y||x-x_n| + |x||y-y_n| + |x-x_n||y-y_n| &< m\delta + \delta^2 = \epsilon \\ |yx - yx_n| + |y-y_n|(|x-0| + |x-x_n|) &< \epsilon \\ |yx - yx_n| + |y-y_n|(|x_n - 0|) &< \epsilon \\ |yx - yx_n| + |yx_n - y_n x_n| &< \epsilon \\ |yx - y_n x_n| &< \epsilon. \end{aligned}$$

Therefore, by our $\epsilon - \delta$ definition of continuity multiplication of real numbers is continuous.

(d) *Division* It is enough to show that the multiplicative inverse map $r : \mathbb{R}/\{0\} \to \mathbb{R}$ defined by $r(x) = 1/x$ is continuous. Then the division map $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ can be written as $v(x,y) = m(i(x), r(y))$ where $i(x)$ is the identity map (continuous by lecture theorem part (b)) and $m(x,y)$ is the multiplication map which we have shown to be continuous. Then by our lemma v is the composition of continuous functions is itself continuous.
Fix any $\epsilon > 0$ and any $x \neq 0$. Let $\delta = \frac{\epsilon|x|^2}{1+\epsilon|x|} > 0$. Consider any $x_n \in B(x,\delta)$. Then

$$\begin{aligned} |x-x_n| &< \delta = \frac{\epsilon|x|^2}{1+\epsilon|x|} \\ (1+\epsilon|x|)|x-x_n| &< \epsilon|x|^2 \\ |x-x_n| + \epsilon|x||x-x_n| &< \epsilon|x|^2 \\ |x-x_n| &< \epsilon|x|(|x| - |x-x_n|). \end{aligned}$$

Applying the triangle inequality we have

$$\begin{aligned}
|x - x_n| &< \epsilon|x|(|x - x_n| + |x_n - 0| - |x - x_n|) \\
|x - x_n| &< \epsilon|x|(|x_n|) \\
\frac{|x - x_n|}{|x||x_n|} &< \epsilon \\
\left|\frac{1}{x_n} - \frac{1}{x}\right| &< \epsilon.
\end{aligned}$$

Therefore, by our $\epsilon - \delta$ definition of continuity taking reciprocals of real numbers is continuous.

□

8. *Given $f, g : X \to \mathbb{R}$ are continuous, prove that $f + g, fg, f - g$, and f/g (provided g is nowhere zero) are all continuous.*

   Notice that these are all functions of the form $h(f(x), g(x))$. For example, given $h(a, b) = a + b$, then $f(x) + g(x) = h(f(x), g(x))$. If we can show that, given continuous $h : \mathbb{R}^2 \to \mathbb{R}$, that $h(f(x), g(x))$ is continuous, then we would be done.

   Let $p : X \to \mathbb{R}^2$ be defined as $p(x) = (f(x), g(x))$. Then we have that $h(f(x), g(x))$ is just $h \circ p(x)$. Since we know compositions of continuous functions are continuous, we just need to show that p is continuous. (See the lemma in problem 7 for the proof of this claim.)

9. In $\mathbb{R}^n$ (and metric spaces, in general), $x_n \to x$ means that given $\epsilon > 0$ there is a finite integer N such that $d(x_n, x) < \epsilon$ for all $n > N$. Show that this agrees with the definition of convergence given for topological spaces.

   Problem rephrased:
   In topology:
   $x_n \to x \Leftrightarrow \forall\, U$ neighborhood of x $\exists\, N \in \mathbb{N}$ such that $n > N \Rightarrow x_n \in U$
   In $\mathbb{R}^d$: $x_n \to x \Leftrightarrow \forall\, \epsilon > 0 \exists\, N \in \mathbb{N}$ such that $n > N \Rightarrow d(x_n, x) < \epsilon$
   Show these statements are equivalent.

   Proof: ($\Leftarrow$) Let $x_n \to x$ in $\mathbb{R}^d$. Let U be a neighborhood of x.
   Then $\exists\, \epsilon > 0$ such that $B(x, \epsilon) \subseteq U$
   So $\exists\, N \in \mathbb{N}$ such that for $n > N$, $d(x_n, x) < \epsilon$ which implies $x_n \in B(x, \epsilon)$
   $\therefore x_n \in U$.
   ($\Rightarrow$) Let $x_n \to x$ (in topology). Given $\epsilon > 0$, $B(x, \epsilon)$ (which is a neighborhood of x) $\exists$ $N \in \mathbb{N}$ such that for $n > N$, $x_n \in B(x, \epsilon)$ which implies $d(x_n, x) < \epsilon$.□

10. (The solution to 10 was not typed up due to a miscommunication)

11. (The solution to 11 was not typed up, because of its similarity to 7)

12. Using the closed set formulation of continuity show that the sets $\{(x,y)|xy = 1\}$, $\{(x,y)|x^2+y^2 = 1\}$, and $\{(x,y)|x^2+y^2 \le 1\}$ are closed in $\mathbb{R}^2$.

Proof. Recall that for $f : X \to Y$ a continuous function, $f^{-1}(C)$ is closed in X whenever C is closed in Y. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = xy$. By exercise 7, f is continuous. $\{1\}$ is a closed set in $\mathbb{R}$. Then $f^{-1}(\{1\}) = \{(x,y)|xy = 1\}$ is closed in $\mathbb{R}^2$. Similarly, let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x,y) = x^2+y^2$. Then $g^{-1}(\{1\}) = \{(x,y)|x^2+y^2 = 1\}$ is closed in $\mathbb{R}^2$. Since $(-\infty,1]$ is a closed set, $g^{-1}((-\infty,1]) = \{(x,y)|x^2+y^2 \le 1\}$ is closed in $\mathbb{R}^2$.

□

13. Let $f_n : \mathbb{R} \to \mathbb{R}$ be defined by $f_n(x) = \frac{1}{n^3(x-(1/n))^2+1}$ and let $f(x) = 0$. Show that $f_n(x) \to f(x)$ for each x, but f_n doesn't converge uniformly to f.

Pointwise Convergence:

Let $\epsilon > 0$ and $x \in \mathbb{R}$ be given. We want to show that $f_n(x)$ converges pointwise to zero. Note that if $x = 0$, then $f_n(0) = \frac{1}{n+1}$, which clearly converges to 0 as $n \to \infty$. When $x \neq 0$, I claim that if $N = \frac{\sqrt{\frac{1}{\epsilon}-1}}{x}$, then for all $n > N$, $f_n(x) < \epsilon$.

Proof of claim:

$$\begin{aligned}
n &> \frac{\sqrt{\frac{1}{\epsilon}-1}}{x} \\
(nx-1)^2 &> 1/\epsilon - 1 \\
n^2x^2 - 2nx + 1 &> 1/\epsilon - 1 \\
n^3x^2 - 2n^2x + n &> 1/\epsilon - 1 && \text{(since } n \ge 1) \\
n^3(x^2 - (2x/n) + (1/n^2)) &> 1/\epsilon - 1 \\
n^3(x-(1/n))^2 &> 1/\epsilon - 1 \\
\frac{1}{n^3(x-(1/n))^2+1} &< \epsilon \\
f_n(x) &< \epsilon.
\end{aligned}$$

Thus, we can make $f_n(x) < \epsilon$, so we have pointwise convergence to 0.

Uniform Convergence:

We see that $f_n(1/n) = 1$ for all n, so for $0 < \epsilon < 1$, there's no N such that for all $x \in \mathbb{R}$ and all $n > N$, $f_n(x) < \epsilon$. Thus, f_n doesn't converge uniformly to 0.

14. *(a) If $\{s_n\}$ is a bounded sequence of real numbers and $s_n \le s_{n+1}$ for each n, then $\{s_n\}$ converges.*

Let T be the least upper bound of $\{s_n\}$. Since $\{s_n\}$ is a bounded sequence, we know $T < \infty$. I claim that the s_n converge to T. Take any ball B of radius ϵ centered around T. B must contain some s_N for some N, because otherwise $T-\epsilon$ would be upper bound of the s_n, contradicting the leastness of T. Since it contains s_N, it also contains s_n for all $n \geq N$, since s_n are monotonically increasing. Therefore, s_n converge to T.

(b) Let $\{a_n\}$ be a sequence of real numbers. Define $s_n = \sum_{i=1}^n a_i$. If $s_n \to s$, we say the infinite series $\sum_{i=1}^\infty a_i$ converges to s. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum ca_i + b_i$ converges to $cs + t$.

Take some $\epsilon > 0$. Note that if $c = 0$, then the result is obvious. Otherwise, if $\sum a_i \to s$, then there exists an N_a such that for $n \geq N_a$, $|\sum_{i=1}^n a_i - s| < \frac{\epsilon}{2|c|}$. Likewise, there exists an N_b such that for $n \geq N_b$, $|\sum_{i=1}^n b_i - t| < \epsilon/2$. Let $N = \max\{N_a, N_b\}$. Therefore, we see that for $n \geq N$,

$$\begin{aligned} \left|\sum_{i=1}^n (ca_i + b_i) - (cs + t)\right| &= \left|c\left(\sum_{i=1}^n a_i - s\right) + \left(\sum_{i=1}^n b_i - t\right)\right| \\ &< |c|\left(\frac{\epsilon}{2|c|}\right) + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore, $\sum ca_i + b_i$ converges to $cs + t$. □

(c) (Comparison test) If $|a_i| \leq b_i$ for each i and $\sum b_i$ converges then $\sum a_i$ converges.

Let $c_i = \begin{cases} a_i & \text{if } a_i \geq 0 \\ 0 & \text{otherwise} \end{cases}$, and let $d_i = \begin{cases} a_i & \text{if } a_i \leq 0 \\ 0 & \text{otherwise} \end{cases}$. Suppose $\sum b_i \to b$. Then we see that both $\sum c_i$ and $\sum d_i$ are bounded by b, and are strictly increasing. Therefore, by part (a), we have both $\sum c_i$ converges, and $\sum d_i$ converges.

Suppose $\sum c_i \to c$ and $\sum d_i \to d$. I claim that $\sum a_i \to c - d$. Take some $\epsilon > 0$. There exists an N_c such that for $n \geq N_c$, $|\sum_{i=1}^n c_i - c| < \epsilon/2$. Likewise, there exists an N_d such that for $n \geq N_d$, $|\sum_{i=1}^n d_i - d| < \epsilon/2$. Let $N = max\{N_a, N_b\}$. Then, we see that for $n \geq N$,

$$\left|\sum_{i=1}^n a_i - (c - d)\right| = \left|\left(\sum_{i=1}^n c_i - c\right) + \left(\sum_{i=1}^n d_i - d\right)\right|.$$

for $\gamma \geq N_c$ and $\delta \geq N_d$. Therefore, we have

$$\begin{aligned} \left|\sum_{i=1}^n a_i - (c - d)\right| &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore, the a_i converge. □

(d) (Weierstrass M-test) Given $f_n : X \to \mathbb{R}$, and let $s_n(x) = \sum_{i=1}^n f_i(x)$. If $f_i(x) \leq b_i$, for all x and i, where $\sum b_i$ converges, then $s_n(x)$ converges uniformly to a function $s(x)$.

Let $s(x) = \sum_{i=1}^{\infty} f_i(x)$ pointwise, which we know converges for each individual x because of part (c). We want to show this convergence is uniform.

Suppose it were not. Then there exists an $\epsilon > 0$ such that for any n, there exists an x such that

$$\left| \sum_{i=1}^{n} f_i(x) - s(x) \right| \geq \epsilon.$$

We know that since $\sum b_i$ converges, it is Cauchy. So, there exists an N such that for $n, k \geq N$,

$$|\sum_{i=n}^{k} b_i| < \epsilon$$

for the same ϵ as above.

For this value of N, there exists an x_0 such that

$$\left| \sum_{i=1}^{N} f_i(x_0) - s(x_0) \right| \geq \epsilon.$$

We know that $\sum_{i=1}^{n} f_i(x_0) \to s(x_0)$. Therefore, there exists a $K > N$, such that for $k \geq K$, we have

$$\left| s(x_0) - \sum_{i=1}^{k} f_i(x_0) \right| < \epsilon/2$$

Therefore, combining the last two inequalities with the triangle inequality, we get

$$\begin{aligned}
\left| \sum_{i=1}^{N} f_i(x_0) - s(x_0) \right| - \left| s(x_0) - \sum_{i=1}^{k} f_i(x_0) \right| &\geq \epsilon - \epsilon/2 \\
\left| \sum_{i=1}^{N} f_i(x_0) - s(x_0) + s(x_0) - \sum_{i=1}^{k} f_i(x_0) \right| &\geq \epsilon/2 \\
\left| \sum_{i=N+1}^{k} f_i(x_0) \right| &\geq \epsilon/2
\end{aligned}$$

However, we see that (using the triangle inequality again)

$$\begin{aligned}
\left| \sum_{i=N+1}^{k} f_i(x_0) \right| &\leq \sum_{i=N+1}^{k} |f_i(x_0)| \\
&\leq \sum_{i=N+1}^{k} b_i \\
&< \epsilon/2
\end{aligned}$$

This is a contradiction. Therefore, the convergence is in fact uniform. □

15. Let f be a uniformly continuous real-valued function on a bounded subset E of the real line. Show that f is bounded on E.

Proof.

Since E is bounded, $E \subset [-M, M]$ for some $M \geq 0$. Let $\varepsilon = 1$. Since f is uniformly continuous, choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon = 1$ whenever $|x - y| < \delta$. Let $\mathcal{B} = \{B(x, \frac{\delta}{2}) | x \in [-M, M]\}$ Since $[-M, M]$ is compact and $\mathcal{B}$ is an open cover of $[-M, M]$, there exists finite subcover $B(x_1, \frac{\delta}{2}), \ldots, B(x_n, \frac{\delta}{2})$. Let $x, y \in E \cap B(x_i, \frac{\delta}{2})$ for some $i = 1, \cdots, n$. Then $|x - y| < \delta$ which implies $|f(x) - f(y)| < 1$. Hence f is bounded on each $E \cap B(x_i, \frac{\delta}{2})$, say by N_i. Since $B(x_1, \frac{\delta}{2}), \ldots, B(x_n, \frac{\delta}{2})$ covers E, f is bounded on E by $\max\{N_i\}$ for $i = 1, \ldots, n$.

Show that f need not be bounded if E is not bounded.

Example:

Let $E = \mathbb{R}$ and let $f(x) = x$. Then f is uniformly continuous, since we are given $\varepsilon > 0$, and letting $\delta = \varepsilon$, $|x - y| < \delta \Rightarrow |x - y| < \varepsilon \Rightarrow |f(x) - f(y)| < \varepsilon$. Finally, f is clearly not bounded on $\mathbb{R}$, since given any $M > 0$, $f(M + 1) = M + 1 > M$.

16. Consider the function f defined by $f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ 1/\text{n} & x = \frac{m}{n} \text{ (m,n rel.prime, } n > 0) \\ 1 & x = 0. \end{cases}$

Prove that f is continuous at every irrational and discontinuous at every rational.

Proof. Let $x_0 \in \mathbb{Q} - \{0\}$ such that $x_0 = \frac{m_0}{n_0}$ where m_0, n_0 are relatively prime and $n_0 > 0$. Fix $\epsilon < \frac{1}{n_0}$. For any $\delta > 0$, there is some $x \in \mathbb{R} - \mathbb{Q}$ such that $x \in (x_0 - \delta, x_0 + \delta)$. So $\mid x_0 - x \mid < \delta$ but $\mid f(x) - f(x_0) \mid = \mid 0 - \frac{1}{n_0} \mid = \frac{1}{n_0} > \epsilon$. So f is discontinuous at $x_0 \in \mathbb{Q} - \{0\}$.

Now consider the case where $x = 0$. Then for $\epsilon < 1$ and any $\delta > 0$, there is some $x \in \mathbb{R} - \mathbb{Q}$ such that $x \in (-\delta, \delta)$. Again $\mid 0 - x \mid < \delta$ but $\mid f(x) - f(0) \mid = \mid 0 - 1 \mid = 1 > \epsilon$. We have f discontinuous at 0, so f is discontinuous for all $x \in \mathbb{Q}$.

Now let $x_0 \in \mathbb{R} - \mathbb{Q}$ and fix $\epsilon > 0$. Let N be the smallest $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. For each $n < N$ define $q_n = \frac{m}{n}$ such that m is the smallest integer such that $\frac{m}{n} > x_0$ and m,n relatively prime. Also define $q_n' = \frac{m}{n}$ such that m is the largest integer such that $\frac{m}{n} < x_0$ and m,n relatively prime. Consider the finite set of rationals $A = \{q_n \mid n < N\} \cup \{q_n' \mid n < N\}$ and let $\delta = min\{d(a, x_0) \mid a \in A\}$.
Then $\mid x - x_0 \mid < \delta$ means $\mid f(x) - f(x_0) \mid = 0 < \epsilon$ if $x \in \mathbb{R} - \mathbb{Q}$. And if $x \in \mathbb{Q}$, then $x = \frac{m'}{n'}$ where m', n' are relatively prime and $n' \geq N$, so $\mid f(x) - f(x_0) \mid = \mid f(x) \mid = \frac{1}{n'} \leq \frac{1}{N} < \epsilon$.

Therefore f continuous at every irrational number.

$\square$

17. Let $f : X \longrightarrow (R)$ be a continuous function on a metric space. Show that the zero set $Z_f = \{x | f(x) = 0\}$ is closed.

    The set $\{0\}$ is closed in $\mathbb{R}$. Therefore its continuous preimage

    $$Z_f = \{x | f(x) = 0\}$$

    is closed.

18. If A is a nonempty subset of a metric space X, define the distance from x to A to be $\delta_A(x) = \text{glb}_{y \in A} d(x, y)$. Prove: (a) $\delta_A(x) = 0 \Leftrightarrow x \in \overline{A}$, and (b) δ_A is uniformly continuous.

    (a) We have that $x \in \overline{A}$ if and only if every open set containing x contains a point of A. But this is true if and only if every open ball about x contains a point of A. By the definition of an ϵ-ball, this is true if and only if for every $\epsilon > 0$, there exists $y \in A$ such that $d(x, y) < \epsilon$. But this is so if and only if $\text{glb}_{y \in A} d(x, y) = 0$, which is equivalent to $\delta_A(x)$.

    (b) Let $\epsilon > 0$ be given. Suppose $x, x' \in X$ with $d(x, x') < \epsilon$; we will show that $|\delta_A(x) - \delta_A(x')| < \epsilon$, establishing uniform continuity.
    Suppose to the contrary that $|\delta_A(x) - \delta_A(x')| > \epsilon$, and assume without loss of generality that $\delta_A(x') > \delta_A(x)$. Then we can write

    $$\delta_A(x') = \delta_A(x) + \epsilon + \gamma,$$

    where $\gamma > 0$. But since $\delta_A(x)$ is the *greatest* lower bound of $\{d(x, z) | z \in A\}$, there exists a $y \in A$ such that $d(x, y) < \delta_A(x) + \gamma$. Then by the triangle inequality,

    $$\begin{aligned} d(x', y) &\leq d(x', x) + d(x, y) \\ &< \epsilon + \delta_A(x) + \gamma \\ &= \delta_A(x'). \end{aligned}$$

    But this contradicts that $\delta_A(x')$ is a lower bound for $\{d(x', z) | z \in A\}$.

19. Let A and B be disjoint nonempty closed subsets of a metric space X. Define $f : X \rightarrow \mathbb{R}$ by

    $$f(x) = \frac{\delta_A(x)}{\delta_A(x) + \delta_B(x)}$$

    for all $x \in X$.

(a) Show f is continuous and the range of f lies in [0,1].

Note δ_A and δ_B are continuous on X, so $\delta_A + \delta_B$ is as well. Therefore, $f = \frac{\delta_A}{\delta_A+\delta_B}$ will be continuous on X provided $\delta_A + \delta_B \neq 0$ on X. But, if there exists some $x \in X$ where $\delta_A(x) + \delta_B(x) = 0$ then $\delta_A(x) = \delta_B(x) = 0$ since both of the δ functions are nonnegative. Thus, by 18 a), $x \in \bar{A} = A$ and $x \in \bar{B} = B$. But this means A and B are not disjoint, contradicting our assumption that they are. Therefore, $\delta_A + \delta_B > 0$ on X which implies f is continuous on X.

To show the range property of f, note $\delta_A \geq 0$ and $\delta_A + \delta_B > 0$ on X which implies $f \geq 0$ on X. Also, if $x \in X$

$$0 \leq f(x) = \frac{\delta_A(x)}{\delta_A(x) + \delta_B(x)} \leq \frac{\delta_A(x) + \delta_B(x)}{\delta_A(x) + \delta_B(x)} = 1$$

using the fact that δ_B is a nonnegative function on X.

(b) Show $f(x) = 0$ iff $x \in A$ and $f(x) = 1$ iff $x \in B$.

Note $f(x) = 0$ iff $\delta_A(x) = 0$ iff $x \in \bar{A} = A$. Also, $f(x) = 1$ iff $\delta_A(x) = \delta_A(x) + \delta_B(x)$ iff $\delta_B(x) = 0$ iff $x \in \bar{B} = B$.

(c) Show that every closed set A in X is the zero set for some continuous function.

If $A = \emptyset$, then choose the function identically 1 on all of X. If $A \neq \emptyset$, then pick δ_A, which vanishes identically on $\bar{A} = A$.

(d) Show that there exist disjoint open sets U and V where $A \subseteq U$ and $B \subseteq V$.

Let f be defined by

$$f(x) = \frac{\delta_A(x)}{\delta_A(x) + \delta_B(x)}$$

on X. Let $\hat{U} = [0, \frac{1}{2})$ and $\hat{V} = (\frac{1}{2}, 1]$ which are open in [0,1]. Thus, $U = f^{-1}(\hat{U})$ and $V = f^{-1}(\hat{V})$ are open in X by the continuity of f. Also, $A = f^{-1}(\{0\}) \subseteq f^{-1}([0, \frac{1}{2})) = f^{-1}(\hat{U}) = U$ and $B = f^{-1}(\{1\}) \subseteq f^{-1}((\frac{1}{2}, 1]) = f^{-1}(\hat{V}) = V$. Finally, because f is a function, U and V are disjoint.

20. Suppose $f, g : X \to Y$ are continuous mappings between metric spaces and E is a dense subspace of X.

a) Prove $f(E)$ is dense in $f(X)$.

We must show $\overline{f(E)} = f(X)$. Since f is continuous, we know $f(\overline{E}) \subseteq \overline{f(E)}$. So $f(\overline{E}) = f(X) \subseteq \overline{f(E)}$. And $f(E) \subseteq f(X)$ implies $\overline{f(E)} \subseteq \overline{f(X)} = f(X)$ since $f(X)$ is closed in $f(X)$. So $\overline{f(E)} = f(X)$ and hence $f(E)$ is dense in $f(X)$.

b) Prove if $f(x) = g(x)$ for all x in E, then $f(x) = g(x)$ for all x in X.

Let $y \in X - E$. We know $y \in \overline{E}$ since $y \in X$. Suppose $f(y) \neq g(y)$. Let $d = d(f(y), g(y))$. So $B(f(y), \frac{d}{3}) \cap B(g(y), \frac{d}{3}) = \emptyset$. Let $g(x_n) = f(x_n)$ be sequences such that $x_n \in E$ and $d(y, x_n) < \frac{1}{n}$. We know these sequences exist since we know $f(x) = g(x)\ \forall x \in E$ and $y \in \overline{E}$. Since f is continuous, $f(x_n) \to f(y)$. So there exists some $N_1 \in \mathbb{N}$ such that $f(x_n) \in B(f(y), \frac{d}{3})\ \forall n > N_1$. Likewise, there is some $N_2 \in \mathbb{N}$ such that $g(x_n) \in B(g(y), \frac{d}{3})\ \forall n > N_2$. Choose m such that $m > N_1$ and $m > N_2$. Then $f(x_m) \in B(f(y), \frac{d}{3})$ and $g(x_m) = f(x_m) \in B(g(y), \frac{d}{3})$. So $f(x_m) \in B(f(y), \frac{d}{3}) \cap B(g(y), \frac{d}{3})$. This is a contradiction, so $f(y) = g(y)$ must hold $\forall y \in X$.

21. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is one-to-one and satisfies $d(x, y) = 1$ implies that $d(f(x), f(y)) = 1$. Show that $d(x, y) = d(f(x), f(y)$ for all $x, y \in \mathbb{R}^2$.

Proof. Fix any point $x \in \mathbb{R}^2$ and any point $y_0 \in \mathbb{R}^2$ such that $d(x, y_0) = 1$. Then define points y_i for $i = 1, 2, 3, 4, 5$ by rotating along the circle of radius 1 centered at x by $\pi i/3$ radians from y_0. Since all of the triangles formed by the segments connecting the y_i and x are equilateral, the y_i are all separated by a distance of 1 unit from y_{i+1} and y_{i-1} (where the subscript addition is mod 6).

Since $d(a, b) = 1$ implies that $d(f(a), f(b)) = 1$, the circle of radius 1 centered at x must map into the circle of radius 1 centered at $f(x)$. In particular, all of the $f(y_i)$ must lie on this circle. Once the location of $f(y_0)$ is fixed on this circle, there are only two points a distance one away from $f(y_0)$ that are still on the circle (each point $\pi/3$ radians away from y_0). Since $d(y_0, y_1) = 1$, y_1 must map to one of these two points, and this then determines where each of the remaining y_i must map.

Consider the perpendicular bisector of y_i and y_{i+1}. There are exactly two points on the perpendicular bisector that are a distance of 1 unit away from y_i and y_{i+1}. One of these points is x, call the other point z. Then $f(z)$ must be one of the two points that is exactly 1 unit away from $f(y_i)$ and $f(y_{i+1})$. One of these points is $f(x)$, and since f is one-to-one the other point must be $f(z)$. Continuing in this manner we see that f must map any equilateral triangular lattice of points separated by 1 unit onto a congruent equilateral triangular lattice of points separated by 1 unit, and that this map is determined by the image of three adjacent points (i.e. any one triangle) of the lattice.

Now consider any point p on the circle of radius 1 centered at x such that $p \neq y_0$. Let θ be the angle from y_0 to p, measured in the direction of y_1. We want to show that $f(p)$ must be the point on the circle of radius 1 centered at $f(x)$ such that the angle from $f(y_0)$ to $f(p)$ measured in the direction of $f(y_1)$ is θ. If we can show this, then we will know that $d(y_0, p) = d(f(y_0), f(p))$. Since x was chosen arbitrarily, y_0 was chosen arbitrarily on the circle of radius 1 centered at x, and p was chosen arbitrarily among points on this circle not equal to y_0, we would know that for all points a, b such that $d(a, b) < 2$, $d(a, b) = d(f(a), f(b))$. To show that $d(a, b) = d(f(a), f(b))$ for the case when $d(a, b) = r \geq 2$, we can repeat the above lattice creation process on a circle of radius $r/\lfloor r \rfloor$. Because of the arbitrary choice of x and y_0, the lattice procedure shows that $d(a,b) = d(f(a), f(b))$ whenever $d(a,b)$ is an integer multiple of the original circle's radius.

Before we prove our claim about $f(p)$, note that for any two points a, b such that $d(a,b) \leq 2$, we can find a point c such that $d(a,c) = d(b,c) = 1$. Then $d(f(a), f(c)) = d(f(b), f(c)) = 1$, so by the triangle inequality $d(f(a), f(b)) \leq 2$.

We know that $f(p)$ must be on the circle of radius 1 centered at $f(x)$. Assume that the angle from $f(y_0)$ to $f(p)$ measured in the direction of $f(y_1)$ is NOT θ. We know that the equilateral triangular lattice of points separated by 1 unit of which p and x are a part must be mapped onto a congruent equilateral triangular lattice of points separated by 1 unit. In particular, each point on the ray $\overrightarrow{xp}$ that is a distance $r \in \mathbb{Z}_{>0}$ from x must be mapped to the point on the ray $\overrightarrow{f(x)f(p)}$ a distance r from $f(x)$. So assume that the angle from $f(y_0)$ to $f(p)$ measured in the direction of $f(y_1)$ is $\alpha \neq \theta$. Let $\beta = |\alpha - \theta|$. Consider the point on $\overrightarrow{xp}$ a distance $r = \lceil \frac{2}{|\sin(\beta/2)|} \rceil$ from x. Call this point a. Let $f(q)$ be the point such that the angle from $f(y_0)$ to $f(q)$ in the direction of $f(y_1)$ is θ, and the distance from $f(q)$ to $f(x)$ is r. Let b be any point on the equilateral triangular lattice of points separated by 1 unit determined by x and y_0 such that $d(a,b) < 2$. Then $f(a)$ must be the point on $\overrightarrow{f(x)f(p)}$ a distance r from $f(x)$, which means that $f(a)$ has been rotated by an angle β about $f(x)$ relative to the image of the equilateral triangular lattice determined by $f(x)$ and $f(y_0)$. This corresponds to a shift of distance $2r\sin(\beta/2) \geq 4$ for the point $f(a)$ from the point $f(q)$, which by the triangle inequality implies that $d(f(a), f(b)) > 2$. But this contradicts our above claim that any pair of points less than or equal to two units apart must map to a pair of points less than or equal to two units apart. Therefore $f(p)$ must be the point on the circle of radius 1 centered at $f(x)$ such that the angle from $f(y_0)$ to $f(p)$ measured in the direction of $f(y_1)$ is θ, completing our proof.

$\square$