Lecture Notes for Analysis Class

Topological Spaces

A *topology* for a set X is a collection $\mathcal{T}$ of subsets of X such that:

(a) X and the empty set are in $\mathcal{T}$.

(b) Unions of elements of $\mathcal{T}$ are in $\mathcal{T}$.

(c) Finite intersections of elements of $\mathcal{T}$ are in $\mathcal{T}$.

A set for which a topology has been specified is called a *topological space*. A topological space is often denoted by the pair $(X, \mathcal{T})$ whenever the specified topology is relevant. The sets in the topology are called the *open sets*. A set E is called *closed* if E^c is open. Notice that X and the empty set are always both open and closed.

Examples

(a) For any set X the set of all subsets of X is a topology. It is called the *discrete topology*. The collection $\{\{\}, X\}$ is also a topology for any set X. It is called the *trivial topology*.

(b) Let $\mathcal{T}$ be the collection of subsets U of X such that $X – U$ is either finite or all of X. Then $\mathcal{T}$ is a topology and it is called the *finite compliment topology*. To see that $\mathcal{T}$ is indeed a topology we need only to check the conditions of the definition. Clearly, X and the empty set are in $\mathcal{T}$ since $X – X$ is empty (therefore, finite) and $X – \{\}$ is X. If $U_\alpha \in \mathcal{T}$ then $X - U_\alpha$ is finite. Since the intersection of finite sets is finite, $X - \bigcup U_\alpha = \bigcap (X - U_\alpha)$ is finite. Thus $\bigcup U_\alpha \in \mathcal{T}$. If $U_i \in \mathcal{T}$ for i = 1, 2,…, n then $X - U_i$ is finite. Since a finite union of finite sets is finite, $X - \bigcap U_i = \bigcup (X - U_i)$ is finite. Thus $\bigcap U_i \in \mathcal{T}$.

(c) Let $X = \{a,b,c\}$ and $\mathcal{T} = \{X, \{\}, \{a\}, \{b,c\}\}$. Then $\mathcal{T}$ is a topology. Note that $\{a\}$ and $\{b,c\}$ are both open and closed.

(d) Let $(X, \mathcal{T})$ be a topological space and Y be a subset of X. The collection $\{Y \cap U \mid U \in \mathcal{T}\}$ is a topology on Y. It is called the *subspace topology*.

Exercises

(1) Check examples (c) and (d).

(2) Is it possible for a set to be open in the subspace topology but not open in the space?

Definitions

(a) A *basis* for a topology on X is a collection $\mathcal{B}$ of subsets of X such that:

 (1) For each $x \in X$, there is at least one B in $\mathcal{B}$ containing x.

 (2) If $x \in B_1 \cap B_2$ ($B_1, B_2 \in \mathcal{B}$) then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

(b) The sets in a basis are called *basis elements*.

(c) A set U is open in the *topology generated by* $\mathcal{B}$ if for each x in U there is a basis element B containing x such that $B \subset U$.
(d) Suppose $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on the same set. If $\mathcal{T}_1 \subset \mathcal{T}_2$, we say that $\mathcal{T}_2$ is *finer* than $\mathcal{T}_1$ and that $\mathcal{T}_1$ is *coarser* than $\mathcal{T}_2$. Note the direction of containment; the finer topology contains more open sets than the coarser topology. If $\mathcal{T}_1 \subset \mathcal{T}_2$ and $\mathcal{T}_1 \neq \mathcal{T}_2$, we say that $\mathcal{T}_2$ is *strictly finer* than $\mathcal{T}_1$ and that $\mathcal{T}_1$ is *strictly coarser* than $\mathcal{T}_2$.
(e) A space is *Hausdorff* if for each pair of points x and y in the space there are disjoint open sets A and B that contain x and y, respectively.

Examples

(a) Suppose $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are topological spaces. Let $\mathcal{B} = \{U \times V \mid U \in \mathcal{T}$ and $V \in \mathcal{T}'\}$. Then $\mathcal{B}$ is a basis for a topology on $X \times Y$. Indeed, every point is in $X \times Y$ which is itself a basis element and the intersection of two basis elements is another basis element ($(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)$). The *product topology* on $X \times Y$ is the topology generated by $\mathcal{B}$.
(b) Suppose $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$, respectively. Then $\mathcal{B} = \{B_1 \times B_2 \mid B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2\}$ is a basis for the product topology on $X \times Y$.
(c) The *standard topology* on R is the one generated by the basis $\{(a,b) \mid a < b\}$. The *standard topology* on R^2 is the product topology of *standard topology* on R with itself. From (b) we see that $\{(a,b) \times (c,d) \mid a < b, c < d\}$ is a basis for the standard topology on R^2.
(d) Any simply ordered set can be given the *order topology*. It is the topology generated by the basis $\{(a,b) \mid a < b\} \cup \{[a_0, b) \mid a_0$ is the smallest element (if it exists)$\}$ $\cup \{(a, b_0] \mid b_0$ is the largest element (if it exists)$\}$. On R the standard topology is the order topology since there is no largest or smallest element. The positive integers Z_+ is an ordered set with a smallest element. The order topology on Z_+ is the discrete topology.

Theorems

(a) Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. The following are equivalent:
   (1) $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.
   (2) For each element x and each basis element B in $\mathcal{B}_1$ containing x, there is a basis element B' in $\mathcal{B}_2$ such that $x \in B' \subset B$.

(b) Suppose that $\mathcal{C}$ is a collection of open sets in a topological space X such that for each open set U and each x in U there is a C in $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis for the topology on X.

Exercises

(3) Show that the collection $\mathcal{B}$ from example (b) is a basis.

(4) Show that the collection $\{(a,b)\times(c,d) \mid a<b, c<d$ and a,b,c,d are rational$\}$ is a basis for the standard topology on R^2.
(5) Show that the collection $\{[a,b) \mid a<b\}$ is a basis on R.
(6) Show that the topology generated by the basis from exercise (5) is strictly finer than the standard topology.
(7) Let $\mathcal{B}$ be a basis. Show that the topology generated by $\mathcal{B}$ equals the collection of all unions of elements from $\mathcal{B}$.
(8) Show that every simply ordered set is Hausdorff in the order topology. Find an example of a non-Hausdorff space.
(9) Find examples that show that an infinite intersection of open sets may be closed and that an infinite union of closed sets may be open.

A function $f: X \to Y$ between topological spaces is called *continuous* if for each open subset O of Y, the set $f^{-1}(O)$ is an open subset of X. We will see that this notion of continuity agrees with the usual notion of continuity from calculus.

Theorem
Let $f: X \to Y$ be a function between topological spaces. Then the following are equivalent:
(a) f is continuous;
(b) $f(\overline{E}) \subset \overline{f(E)}$ for every subset E of X;
(c) $f^{-1}(C)$ is closed in X whenever C is closed in Y.

Theorem (construction of continuous functions)
Let X, Y and Z be topological spaces.
(a) The constant function $f(x) = y_0 \in Y$ is continuous.
(b) If X is a subspace of Y then the inclusion map $f(x) = x \in Y$ is continuous.
(c) If $f: X \to Y$ and $g: Y \to Z$ are continuous then $g \circ f: X \to Z$ is continuous.
(d) If $f: X \to Y$ is continuous and A is a subspace of X then the restricted function $f \mid A: A \to Y$ is continuous.
(e) If $f: X \to Y$ is continuous then a function obtained from f by restricting or expanding the range is continuous.
(f) The map $f: X \to Y$ is continuous if X can be written as $\bigcup_\alpha E_\alpha$ where each E_α is open and $f \mid E_\alpha$ is continuous for each α.
(g) The map $f: X \to Y$ is continuous if for each $x \in X$ and each neighborhood V of $f(x)$ there is a neighborhood U of x such that $f(U) \subset V$. This is called *continuous at x*.
(h) Let $X = A \cup B$, where A and B are closed; let $f: A \to Y$ and $g: B \to Y$ be continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Then the function $h: X \to Y$ defined by $h(x) = f(x)$ for $x \in A$ and $h(x) = g(x)$ for $x \in B$ is continuous.

(i) Let $f : A \to X \times Y$ be given by the equation $f(a) = (f_1(a), f_2(a))$. Then *f* is continuous iff $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous.

Definitions

(a) A bijective function $f : X \to Y$ is called a *homeomorphism* if it and its inverse are continuous.

(b) A mapping $f : X \to Y$ is called an *imbedding* if its restriction $f : X \to f(X)$ is a homeomorphism.

(c) An *open cover* of a subset *E* of *X* is a collection of open sets $\{E_\alpha\}$ from *X* such that $E \subset \bigcup_\alpha E_\alpha$.

(d) A set *K* is *compact* if every open cover admits a finite subcover. More explicitly, if $\{E_\alpha\}$ is any open cover of a compact set *K* then there is a finite number of sets $E_1, ..., E_k$ from $\{E_\alpha\}$ such that $E \subset E_1 \cup ... \cup E_k$.

(e) Two disjoint nonempty open subsets *A* and *B* of *X* are a *separation* of *X* if $A \cup B = X$. If there is no separation of *X* then it is called *connected*.

(f) The *interior* of a set *A* is the largest open set contained in *A*. It is denoted by A^o or Int (*A*).

(g) The *closure* of a set *A* is the smallest closed set that contains *A*. It is denoted by $\overline{A}$.

(h) A point *x* is a *limit point* of a set *A* if every open set containing *x* intersects *A* at some point other than *x*. The set of all limit points of A is denoted by A'.

(i) A *neighborhood* of *x* is any set that contains an open set containing *x*.

(j) A sequence $\{x_n\}$ of points *converges* to a point *x* if for every neighborhood *U* of *x* there is a positive integer *N* such that $x_i \in U$ for all $i > N$. If $\{x_n\}$ does not converge, it *diverges*.

(k) The boundary of a set *A* is $\overline{A} \cap \overline{A^c}$. It is denoted by Bd (*A*).

Exercises

(10) Show that $(X_1 \times X_2 \times ... \times X_{n-1}) \times X_n$ is homeomorphic to $X_1 \times X_2 \times ... \times X_n$.

(11) Show that the interior of *A* is the union of all open subsets of *A*. Show that the closure of *A* is the intersection of all closed sets containing *A*.

(12) Show that $\overline{A} = A \cup A'$.

(13) Show that a set *A* is closed iff $\overline{A} = A$ and that A is closed iff $A' \subset A$.

(14) Show that A^o and Bd (*A*) are disjoint and that $\overline{A} = A^o \cup$ Bd (A).

(15) If *p* is a limit point of *E* (in a Hausdorff space) then every neighborhood of *p* contains infinitely many points of *E*.

(16) A finite point set in a Hausdorff space has no limit points.

(17) Show that if there is a sequence of points from *A* converging to *x* then *x* is a limit point of *A*.

Theorems (connectedness)

(a) X is connected iff the only sets that are both open and closed are the empty set and X itself.
(b) Another formulation of a separation of X is a pair of nonempty sets A and B such that $A \cup B = X$ and $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.
(c) Suppose Y is a connected subspace of X. If A and B form a separation of X then Y is entirely in either A or B.
(d) The union of a collection of connected sets with a point in common is connected.
(e) Let A be connected. If $A \subset B \subset \overline{A}$ then B is also connected.
(f) The image of a connected set under a continuous map is connected.
(g) If X_i is connected for $i = 1,\ldots,n$ then the product $\prod_{i=1}^{n} X_i$ is connected.
(h) A subset E of the real line is connected iff it has the following property: If $x, y \in E$ and $x < z < y$, then $z \in E$.
(i) (Intermediate Value Theorem) Let $f : X \to Y$ be a continuous map of a connected space X into an ordered space Y (with the order topology). If a and b are two points of X and r is a point of Y lying between $f(a)$ and $f(b)$ then there is a point c in X such that $f(c) = r$.

Theorems (compactness)

(a) Every closed subset of a compact set is compact.
(b) If K is a compact set in a Hausdorff space and x is not in K then there exist disjoint open sets A and B such that $K \subset A$ and $x \in B$.
(c) Every compact subset of a Hausdorff space is closed.
(d) The image of a compact space under a continuous map is compact.
(e) Suppose X is compact and Y is Hausdorff. If $f : X \to Y$ is a continuous bijection then f is a homeomorphism.
(f) The product of finitely many compact spaces is compact.
(g) Let X be a space with the order topology and the least upper bound property. Each closed interval in X is compact.
(h) Let X be a nonempty compact Hausdorff space. If every point of X is a limit point of X then X is uncountable.

Metric Spaces

A set X is called a *metric space* if for any two points p and q of X there is an associated number $d(p,q)$, called the distance from p to q, such that

(a) $d(p,q) \geq 0$;
(b) $d(p,q) = 0$ iff $p = q$;
(c) $d(p,q) = d(q,p)$;
(d) $d(p,q) \leq d(p,r) + d(r,q)$ for any r in X.

Any function with these properties is called a *distance function* or a *metric*.

Definitions

(a) The set $B_d(x,\varepsilon)=\{y \mid d(x,y)<\varepsilon\}$ is called the *open ball* of radius ε centered at *x*. Similarly, the set $C_d(x,\varepsilon)=\{y \mid d(x,y)\le\varepsilon\}$ is called the *closed ball* of radius ε centered at *x*. When no confusion will arise the metric *d* will be omitted from the notation.
(b) A set *S* is *bounded* if for every pair of points *x* and *y* in *S* there is a finite number *M* such that $d(x,y)\le M$. A bounded set has a *diameter* that is defined to be the least upper bound of the set $\{ d(x,y) \mid x,y\in S \}$.
(c) Let $f_n : X \to Y$ be a sequence of functions from a set *X* to a metric space *Y*. We say $\{ f_n \}$ *converges uniformly* to $f : X \to Y$ if given $\varepsilon>0$ there is an integer *N* such that $d(f_n(x), f(x))<\varepsilon$ for all $n > N$ and all *x* in *X*.

Examples

(a) The usual distance function $d(p,q) = [\sum_{i=1}^{n}(p_i - q_i)^2]^{1/2}$ (also called the Euclidean metric) turns R^n into a metric space.
(b) Another metric on the Euclidean spaces is given by $d(p,q) = \max\{|p_i - q_i| \text{ for } i = 1,\ldots,n\}$. This called the square metric.
(c) Let *X* be any set and $d(p,q)$ be 1 if $p \neq q$ (and, of course, 0 if $p = q$). Then *d* is a metric (called the discrete metric) and *X* is a metric space.
(d) Any subset *Y* of a metric space *X* is a metric space using the same distance function.

If *X* has a metric *d* defined on it then the metric determines a topology (the metric topology) on *X*. The collection of all open balls is a basis. This basis generates the metric topology on *X*. All metric spaces are Hausdorff. Different metrics may generate the same topology.

Example

We show the Euclidean and square metrics on R^n generate the same topology.

Let *d* and δ denote the Euclidean and square metrics, respectively. It is easy to see that $\delta(x,y)\le d(x,y)\le\sqrt{n}\,\delta(x,y)$ holds for any points *x* and *y*. The first inequality shows that $B_d(x,\varepsilon)\subset B_\delta(x,\varepsilon)$ and so the Euclidean metric topology is finer than the square metric topology. The second inequality shows that $B_\delta(x,\varepsilon/\sqrt{n})\subset B_d(x,\varepsilon)$ and so the square metric topology is finer than then Euclidean metric topology. Thus they are equivalent. The topology generated by both of these metrics is the same as the product topology on R^n.

Exercises

(1) Show that the square metric topology is equivalent to the product topology on R^n.

(2) If d is a metric, show that $\overline{d}(x, y) = \min\{d(x, y), 1\}$ is also a metric (called the *standard bounded metric* associated to d). Show that all sets are bounded using $\overline{d}$.
(3) Show that in a Hausdorff space a convergent sequence must converge to only one point.

Example
We show that the calculus and topological notions of continuity agree on the real line.

Suppose $f : R \to R$ is continuous in the sense that given $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Let O be open and $x_0 \in f^{-1}(O)$. Since f is continuous, x being within δ of x_0 insures that $f(x)$ is within ε of $f(x_0)$. Choose ε such that $\{f(x) : |f(x) - f(x_0)| < \varepsilon\} \subset O$. Then $(x_0 + \delta, x_0 - \delta) \subset f^{-1}(O)$. Thus $f^{-1}(O)$ is open.

Now suppose that $f : R \to R$ is continuous in the sense that $f^{-1}(O)$ is open whenever O is open. Then if $O = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, $f^{-1}(O)$ is open. Thus $f^{-1}(O)$ contains an open ball of some radius $\delta > 0$ centered at x_0. Since $(x_0 - \delta, x_0 + \delta) \subset f^{-1}(O)$, $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \varepsilon$.

Theorems
(a) Let X and Y be metric spaces with metrics d_X and d_Y, respectively. The continuity of $f : X \to Y$ is equivalent to the requirement that given x and $\varepsilon > 0$, there is a $\delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$.
(b) If X is a metric space then $x \in \overline{A}$ iff there is a sequence of points in A that converges to x.
(c) Suppose X is a metric space. A function $f : X \to Y$ is continuous iff $x_n \to x$ in X implies that $f(x_n) \to f(x)$ in Y.
(d) (Uniform Limit Theorem) If $\{ f_n \}$ is a sequence of continuous functions (from a topological space into a metric space) that converges uniformly to f then f is continuous.
(e) A subset of R^n is compact iff it is closed and bounded in the Euclidean or square metric.

Definitions
(a) A space X is *limit point compact* if every infinite subset of X has a limit point.
(b) A space X is *sequentially compact* if every sequence in X has a convergent subsequence.
(c) A function $f : X \to Y$ between metric spaces is *uniformly continuous* if given $\varepsilon > 0$, there is a $\delta > 0$ such that for any two points a and b of X, $d_X(a,b) < \delta \Rightarrow d_Y(f(a), f(b)) < \varepsilon$.

Theorems

(a) If E is an infinite subset of a compact set K then E has a limit point in K (i.e., compactness $\Rightarrow$ limit point compactness).
(b) Limit point compactness implies sequential compactness in a metric space.
(c) Let $\mathcal{A}$ be an open cover of a sequentially compact metric space X. There is a $\delta > 0$ such that for each subset of X having diameter less than δ there is an element of $\mathcal{A}$ containing it.
(d) (Uniform Continuity Theorem) Let $f : X \to Y$ be a continuous map from a compact metric space to a metric space. Then f is uniformly continuous.
(e) In a metric space X, compactness, limit point compactness, and sequential compactness are equivalent.

Definitions

(a) A sequence $\{x_n\}$ of points in a metric space (X, d) is said to be a *Cauchy sequence* if given $\varepsilon > 0$, there is an integer N such that $d(x_n, x_m) < \varepsilon$ whenever $n, m > N$.
(b) A metric space is *complete* if every Cauchy sequence in the space converges to a point in the space.

Exercises

(4) Is every convergent sequence a Cauchy sequence?
(5) Is a closed set of a complete space complete? (If so, in what metric?)
(6) If a space is complete under d, is it complete under $\bar{d}(x, y) = \min\{d(x, y), 1\}$?

Theorems

(a) A metric space X is complete iff every Cauchy sequence in X has a convergent subsequence.
(b) Euclidean space R^n is complete in the Euclidean and square metrics.

Let (Y, d) be a metric space and $\bar{d}$ be the standard bounded metric associated to d. The collection of all functions from some set X into Y (denoted Y^X) can be turned into a metric space. The formula $\bar{\rho}(f, g) = \text{lub}\{\bar{d}(f(x), g(x)) \mid x \in X\}$ defines the *uniform metric* $\bar{\rho}$ on Y^X corresponding to d. If X is a topological space then we may consider the collection $\mathcal{C}(X,Y)$ of all continuous functions from X into Y.

Theorems

(a) If (Y,d) is complete then $(Y^X, \bar{\rho})$ is complete.
(b) Under the uniform metric, $\mathcal{C}(X,Y)$ is closed in Y^X. Thus, if (Y,d) is complete then $(\mathcal{C}(X,Y), \bar{\rho})$ is complete.
(c) There exists a continuous surjection $f: [0,1] \to [0,1]^2$.

Baire Spaces

A space X is said to be a *Baire space* if given any countable collection $\{C_n\}$ of closed sets in X each having empty interior then $\bigcup C_n$ also has empty interior. This definition may also be formulated in terms of open sets: A space X is said to be a *Baire space* if given any countable collection $\{U_n\}$ of open sets in X each being dense in X then $\bigcap U_n$ is also dense in X.

Exercises

(1) Show that these two definitions are equivalent.
(2) Show that every non-empty open set in a Baire space is not the countable union of closed sets having empty interiors.

The following statement is known as *The Baire Category Theorem*.

Every non-empty complete metric space is a Baire Space.

This is a very important theorem in Analysis. It has a number of consequences such as the Principle of Uniform Boundedness, the Open Mapping Theorem, the Closed Graph Theorem, the Inverse Mapping Theorem, and the existence of a continuous nowhere differentiable function.

A subset $\mathcal{F}$ of $\mathcal{C}(X,R)$ is said to be *uniformly bounded* on a subset U of X if there is a positive integer M such that $|f(x)| < M$ for all x in U and for all f in $\mathcal{F}$. Note: $\mathcal{F}$ being bounded in $\rho(f,g) = \text{lub}\,\{|f(x) - g(x)| : x \in X\}$ is equivalent to $\mathcal{F}$ being uniformly bounded on X.

Theorems

(a) (Principle of Uniform Boundedness) Let X be a complete metric space and $\mathcal{F}$ be a subset of $\mathcal{C}(X,R)$ such that for each x in X the set $\mathcal{F}_x = \{f(x) \mid f \text{ in } \mathcal{F}\}$ is bounded. Then there is a nonempty open set U in X on which $\mathcal{F}$ is uniformly bounded.
(b) Let $h:[0,1] \to R$ be a continuous function. Given $\varepsilon > 0$ there is a continuous nowhere differentiable function $g:[0,1] \to R$ such that $|h(x) - g(x)| < \varepsilon$ for all x.

A set that can be written as a countable intersection of open sets is called a G_δ set.

Examples

(1) Every singleton $\{x\}$ in R^n is a G_δ since $\{x\} = \bigcap_{n=1}^{\infty} B(x, 1/n)$.
(2) Let $f : R \to R$ be an arbitrary function. The set, A, consisting of points at which f is continuous is a G_δ. To see this let C_n be the collection of all open sets U such that the diameter of $f(U)$ is less than $1/n$. Set U_n to be the union of all sets in C_n. Then U_n is open and $A = \bigcap U_n$. To see this equality we show double

containment. If $x \in A$ then for every neighborhood $V = (f(x) - \varepsilon, f(x) + \varepsilon)$ of $f(x)$ there is a neighborhood $O = (x - \delta, x + \delta)$ of x such that $f(O) \subset V$. We show that $x \in U_n \forall n$. Given n, pick $V = (f(x) - (4n)^{-1}, f(x) + (4n)^{-1})$. Then $\exists$ a neighborhood O_n of x such that $f(O_n) \subset V \Rightarrow$ diameter($f(O_n)) < (2n)^{-1} < 1/n$. Thus $x \in O_n \subset U_n \forall n$. (On the other hand) If $x \in \bigcap U_n$ then $\forall n$ there exists a neighborhood O_n of x such that diameter ($f(O_n)) < 1/n$. Given a neighborhood $V = (f(x) - \varepsilon, f(x) + \varepsilon)$ of $f(x)$, pick $n > 1/\varepsilon$. Then diameter($f(O_n)) < \varepsilon$. Thus $f(O_n) \subset V \Rightarrow x \in A$.

Is there a function $f: R \to R$ that is continuous precisely on Q (the rationals)? No, because Q is not a G_δ. To see this suppose Q is a G_δ, i.e., $Q = \bigcap_n W_n$ with each W_n open. If $V_q = R - \{q\}$ (for q in Q) then the collection $\mathcal{a}$ = { W_n} $\cup$ { V_q }is a countable collection of dense open sets. Then, by the Baire Category Theorem, the intersection A of all sets in $\mathcal{a}$ is also dense. If x is in A then x is in W_n for each n and V_q for each q. But this implies x is both in and not in Q. Thus A is empty and we have a contradiction.

Is there a function $f: R \to R$ that is continuous precisely on the irrationals? Yes. (See problem 16 from the metric space homework.) This shows that the irrationals are a G_δ.

Exercises

(3) The Baire Category Theorem implies that R cannot be written as the countable union of closed sets having empty interiors. Show that this fails if the sets are not required to be closed.
(4) Show that the rationals are not Baire.
(5) Show that every open subset of a Baire space is a Baire space.