Homework Problems on Topological Spaces

1. Prove DeMorgan's laws: $\left(\bigcup_{\alpha} A_{\alpha}\right)^{c} = \bigcap_{\alpha} A_{\alpha}^{c}$ and $\left(\bigcap_{\alpha} A_{\alpha}\right)^{c} = \bigcup_{\alpha} A_{\alpha}^{c}$.

2. A) Let $\mathcal{T}_c$ be the collection of all subsets U of X such that $X - U$ is countable or all of X. Show that $\mathcal{T}_c$ is a topology on X.
   B) Is the collection of all subsets U of X such that $X - U$ is infinite, empty, or all of X a topology?

3. Suppose $\{\mathcal{T}_i\}$ is a collection of topologies on X.
   A) Show that the intersection of all the $\mathcal{T}_i$ is a topology.
   B) Is the union of all the $\mathcal{T}_i$ a topology?
   C) Show that there is a unique smallest topology containing all the $\mathcal{T}_i$ and a unique largest topology contained in all of the $\mathcal{T}_i$.

4. Let $X = \{a,b,c\}$, $\tau = \{X,\{\},\{a\},\{a,b\}\}$ and $\tau' = \{X,\{\},\{a\},\{b,c\}\}$.
   A) Find the smallest topology containing τ and τ'.
   B) Find the largest topology contained in τ and τ'.

5. Let X be a topological space. Show that the following conditions hold:
   A) The empty set and X are closed.
   B) Arbitrary intersections of closed sets are closed.
   C) Finite unions of closed sets are closed.

6. Prove the following:
   A) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
   B) $\overline{\bigcup A_{\alpha}} \supset \bigcup \overline{A_{\alpha}}$ (Give an example where equality fails.)

7. Let τ and τ' be two topologies on a set X and let i: $(X, \tau') \rightarrow (X, \tau)$ be the identity map.
   A) τ' is finer than $\tau \Leftrightarrow i$ is continuous.
   B) $\tau' = \tau \Leftrightarrow i$ is a homeomorphism.

8. Let $\mathcal{T}_n$ be the topology on the real line generated by the usual basis plus $\{n\}$. Show that $(R, \mathcal{T}_1)$ and $(R, \mathcal{T}_2)$ are homeomorphic, but that $\mathcal{T}_1$ does not equal $\mathcal{T}_2$.

9. Find a function from R to R that is continuous at precisely one point.

10. Show that if $X \subset Y \subset Z$ then the subspace topology on X as a subspace on Y is the same as the subspace topology on X as a subspace of Z.

11. Show that the countable collection $\{(a,b) \times (c,d) \mid a<b$ and $c<d$ and a,b,c,d are rational$\}$ is a basis for R^2.

12. Determine which of the following equations hold. If not, determine whether any inclusion holds.
    A) $\overline{A \cap B} = \overline{A} \cap \overline{B}$
    B) $\overline{\bigcap A_\alpha} = \bigcap \overline{A}_\alpha$
    C) $\overline{A - B} = \overline{A} - \overline{B}$
    D) $(A \cup B)' = A' \cup B'$
    E) $(A \cap B)' = A' \cap B'$

13. If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, what does the connectedness of X in one topology imply about the connectedness of X in the other?

14. Let $\{A_n\}$ be a sequence of connected sets such that A_n intersects A_{n+1} nontrivially for each n. Show that $\bigcup A_n$ is connected.

15. Show that if X is an infinite set then it is connected in the finite complement topology.

16. If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, what does the compactness of X in one topology imply about the compactness of X in the other?

17. Show the following:
    (a) Bd (A) is empty iff A is both open and closed.
    (b) A is open iff Bd (A) = $\overline{A} - A$.

18. For any subset A of the real line (with the usual topology) there are at most 14 sets (including A) that can be formed by using complementation and closure. Prove this by completing the following steps:
    (a) Show that if A is open then $\overline{A} = A^{-c-c-}$.
    (b) Let $\mathbb{K} = \{A, \overline{A}, A^{-c}, A^{-c-}, A^{-c-c}, A^{-c-c-}, A^{-c-c-c},$
    $A^{c}, A^{c-}, A^{c-c}, A^{c-c-}, A^{c-c-c}, A^{c-c-c-}, A^{c-c-c-c}\}$.
    Show that $\mathbb{K}$ is closed under complementation and closure.
    (c) Show that there is an A such that $\mathbb{K}$ has exactly 14 distinct elements.

19. For any subset A of the real line (with the usual topology) there are at most 7 sets (including A) that can be formed by using the interior and closure operations. Prove this by completing the following steps:
    (a) Show that $A^{o-} = A^{o-o-}$ and $A^{-o} = A^{-o-o}$.
    (b) Let $\mathbb{K} = \{A, \overline{A}, A^{-o}, A^{-o-}, A^{o}, A^{o-}, A^{o-o}\}$.
    Show that $\mathbb{K}$ is closed under the interior and closure operations.
    (c) Show that there is an A such that $\mathbb{K}$ has exactly 7 distinct elements.

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20. Show that the product of two Hausdorff spaces is Hausdorff.

21. Prove the Extreme Value Theorem: Let $f : X \to Y$ be a continuous map of a compact space X into an ordered space Y (with the order topology). Then there are points a and b in X such that $f(a) \le f(x) \le f(b)$ for every x in X.

22. Prove the following: A topological space X is compact iff every collection $\{C_\alpha\}$ of closed subsets of X with the property that the intersection of any finite subcollection is nonempty also has the property that $\bigcap C_\alpha$ is nonempty.

Homework Problems on Metric Spaces

1. In R^n, define $d(x, y) = |x_1 - y_1| + ... + |x_n - y_n|$. Show that d is a metric that induces the usual topology. Sketch the basis elements when $n = 2$.

2. In R^n, for $p \ge 1$ define $d(x, y) = \sum_{i=1}^{n} (|x_i - y_i|^p)^{1/p}$. Assume that d is a metric. Show that it induces the usual topology.

3. Show that the topology induced by a metric is the coarsest topology relative to which the metric is continuous.

4. Let d be a metric. Show that $d'(x, y) = d(x, y)/(1 + d(x, y))$ is a bounded metric.

5. Let d be a metric. Show that $\bar{d}(x, y) = \min\{d(x, y), 1\}$ induces the same topology as d.

6. For x and y in R^n, let $x \cdot y = \sum x_i y_i$ and $\|x\| = \sqrt{x \cdot x}$. Show that the Euclidean metric d on R^n is a metric by completing the following:
   (a) Show that $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$.
   (b) Show that $|x \cdot y| \le \|x\| \, \|y\|$.
   (c) Show that $\|x + y\| \le \|x\| + \|y\|$.
   (d) Verify that d is a metric.

7. Prove the continuity of the algebraic operations on the real line. (Hint: For multiplication, show $|x_n y_n - xy| \le |x| \, |y_n - y| + |y| \, |x_n - x| + |x_n - x| \, |y_n - y|$; for division, show first that taking reciprocals is continuous.)

8. Prove: If X is a topological space and $f, g : X \to R$ are continuous then $f + g$, $f - g$ and $f \cdot g$ are continuous. If $g(x) \ne 0$ for all x then f / g is continuous.

9. In R^n (and metric spaces, in general), $x_n \to x$ means that given $\varepsilon > 0$ there is a finite integer N such that $d(x_n, x) < \varepsilon$ for all $n > N$. Show that this agrees with the definition of convergence given for topological spaces.

10. Suppose $f : X \to Y$ is a map between metric spaces that has the property $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$ for all points x_1 and x_2 in X. Show that f is an imbedding. It is called an *isometric imbedding*.

11. Prove: If $x_n \to x$ and $y_n \to y$ in R then $x_n + y_n \to x + y$, $x_n - y_n \to x - y$, $x_n y_n \to xy$, and (provided $y_n, y \neq 0$) $x_n / y_n \to x / y$. (Suggestion: Show $x_n \times y_n \to x \times y$ and use the results of problem 7.)

12. Using the closed set formulation of continuity show that the sets $\{(x, y) \mid xy = 1\}$, $\{(x, y) \mid x^2 + y^2 = 1\}$ and $\{(x, y) \mid x^2 + y^2 \leq 1\}$ are closed in R^2.

13. Let $f_n : R \to R$ be defined by $f_n(x) = \dfrac{1}{n^3[x - (1/n)]^2 + 1}$ and let $f(x) = 0$. Show that $f_n(x) \to f(x)$ for each x, but f_n does not converge uniformly to f.

14. Prove the following:
   (a) If $\{s_n\}$ is a bounded sequence of real numbers and $s_n \leq s_{n+1}$ for each n, then $\{s_n\}$ converges.
   (b) Let $\{a_n\}$ be a sequence of real numbers. Define $s_n = \sum_{i=1}^{n} a_i$. If $s_n \to s$, we say that the infinite series $\sum_{i=1}^{\infty} a_i$ converges to s. Show that if $\sum a_i$ converges to s and $\sum b_i$ converges to t, then $\sum ca_i + b_i$ converges to $cs + t$.
   (c) (Comparison test) If $|a_i| \leq b_i$ for each i and $\sum b_i$ converges then $\sum a_i$ converges. [Hint: First show that $\sum |a_i|$ converges. Then figure out how to use part (b).]
   (d) (Weierstrass M-test) Given $f_n : X \to R$, let $s_n(x) = \sum_{i=1}^{n} f_i(x)$. If $|f_i(x)| \leq b_i$ for all x and i where $\sum b_i$ converges, then $s_n(x)$ converges uniformly to a function $s(x)$. [Hint: Let $r_n = \sum_{i=n+1}^{\infty} b_i$ and show that for $k > n$, $|s_k(x) - s_n(x)| \leq r_n$.]

15. Let f be a uniformly continuous real-valued function on a bounded subset E of the real line. Show that f is bounded on E. Show that f need not be bounded if E is not bounded.

16. Consider the function $f(x)=\begin{cases}0 & x\notin Q\\ 1/n \text{ when } & x=m/n\\ 1 & x=0\end{cases}$ (here m and n are relatively prime and $n > 0$). Prove that f is continuous at every irrational and discontinuous at every rational.

17. Let $f: X \to R$ be a continuous function on a metric space. Show that the zero set $Z_f = \{ x \mid f(x) = 0 \}$ is closed.

18. If A is a nonempty subset of a metric space X, define the distance from x to A to be $\delta_A(x) = \text{glb}_{y\in A}\, d(x,y)$. Prove:
(a) $\delta_A(x) = 0 \Leftrightarrow x \in \overline{A}$.
(b) δ_A is uniformly continuous.

19. Let A and B be disjoint nonempty closed subsets of a metric space X. Define $f(x) = \dfrac{\delta_A(x)}{\delta_A(x)+\delta_B(x)}$. Prove:
(a) f is a continuous function whose range lies in [0,1].
(b) f is 0 precisely on A and 1 precisely on B.
(c) Every closed set in X is the zero set Z_f for some continuous function.
(d) Show that there exists disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

20. A subset E of X is called *dense* if $\overline{E} = X$. Suppose $f, g: X \to Y$ are continuous mappings between metric spaces and that E is a dense subspace of X. Prove:
(a) $f(E)$ is dense in $f(X)$.
(b) If $f(x) = g(x)$ for all x in E then $f(x) = g(x)$ for all x in X.

21. Suppose $f: R^2 \to R^2$ is one-to-one and satisfies $d(x,y)=1$ implies that $d(f(x), f(y)) = 1$. Show that $d(x,y) = d(f(x), f(y))$ for all x and y.

Additional Homework Problems

1. Let $\mathcal{F}$ be a subset of Y^X such that the set $\{d(f(x), g(x)) \mid x \in X\}$ is bounded $\forall f, g \in \mathcal{F}$. Show that $\rho(f, g) = \text{lub}\,\{d(f(x), g(x)) \mid x \in X\}$ is a metric on $\mathcal{F}$ and that $\overline{\rho}$ = min $\{\rho(f, g), 1\}$.

2. (A) Show that there is a continuous surjection $f : [0,1] \to [0,1]^n$ for any positive integer *n*. [Hint: Consider $f \times f$.]
   (B) Is there a continuous surjection from [0,1] to R^2?

3. A *norm* on a vector space *X* is a mapping that assigns to each vector *x* a real number $||x||$ such that:
   (1) $||x|| \geq 0$ and $||x|| = 0$ iff $x = 0$,
   (2) $||x + y|| \leq ||x|| + ||y||$ and
   (3) $||cx|| = |c|\,||x||$ for any constant c.
   A complete normed vector space is called a *Banach* space.
   (A) Show that a norm on *X* can be used to define a metric on *X*.
   (B) Find a metric that is not determined by any norm.
   (C) Consider the space l_∞ of all bounded sequences $x = \{x_n\}$ and let $||x|| = \text{lub}\,\{|x_n|\}$. Show that l_∞ is a Banach space.

4. Assume $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, $1 \leq p < \infty$, $||x||_p = (\sum_1^n |x_i|^p)^{1/p}$, and $1/p + 1/q = 1$.
   (A) Show that $a, b \geq 0 \Rightarrow a^{1/p} b^{1/q} \leq a/p + b/q$. [Hint: For 0< *k* <1 consider the functions $f(t) = t^k$, $g(t) = kt + (1-k)$ when $t \geq 1$.]
   (B) Prove *Holder's inequality*: $\sum_1^n |x_i y_i| \leq ||x||_p\,||y||_q$.
   (C) Prove *Minkowski's inequality*: $||x + y||_p \leq ||x||_p + ||y||_p$.
   (D) Show that $x \mapsto ||x||_p$ is a norm. This shows that the metric from problem 2 is indeed a metric.
   (E) Show that the space l_p^n of all *n* –tuples $x = (x_1, \ldots, x_n)$ with norm $x \mapsto ||x||_p$ is a Banach space.
   (F) Compare basic open balls in R^2 for different values of *p*.
   (G) Do Holder's and Minkowski's inequalities extend to the case $n = \infty$?

5. A (real) *inner product* on a vector space *X* is a map $\langle \cdot,\cdot \rangle : X \times X \to R : (x, y) \mapsto \langle x, y \rangle$ that satisfies:
   (1) $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$,
   (2) $\langle x, y \rangle = \langle y, x \rangle$ and
   (3) $\langle x, x \rangle \geq 0$ with equality occurring iff $x = 0$.
   A complete inner product space is called a *Hilbert space*.

(A) Prove the *Schwarz inequality*: $|\langle x, y\rangle| \leq \|x\| \|y\|$.

(B) Show that $\|x\| = \sqrt{\langle x, x\rangle}$ defines a norm.

(C) Prove the *Parallelogram law*: $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

(D) Show that the Parallelogram law is not true in l_1^2. What can you conclude?

(E) Show that $l_2 = \{x = \{x_i\} \mid (\sum_1^\infty |x_i|^2)^{1/2} < \infty\}$ is a Hilbert space.

6. Let X and Y be metric spaces. Suppose A is a subset of X and Y is complete. Show that if $f : A \to Y$ is uniformly continuous then it can be extended uniquely to a uniformly continuous function $g : \overline{A} \to Y$.