Student Presentations -- Examples

Each of these sets is from a paper dealing with the unique midpoint property. The goal is to add to our library of examples and to see what properties the sets or the medial maps have. Depending on your background, you may not yet fully understand the examples. The main point now is to understand the set and metric described.

Form groups; each group focusing on one presentation. The groups will present their example in an analysis lecture (Friday, June 30 or Monday, July 3?).

Please describe the set and the metric used. Verify that the UMP is (or isn't) present. (This should at least be a geometrical argument.)

Presentation 1: The example in the Ohta/Ono paper, page 219.

Presentation 2: Study example (3.2) from Sam B. Nadler, Jr., An embedding theorem for certain spaces with an equidistant property, Proc. Amer. Math. Soc. **59**, (1976), no. 1, 179-183. Note that UEP (unique equidistant property) is Nadler's notation for UMP.

(3.2) Example. We use polar coordinates. For each $n = 1, 2, \ldots$, let

$$A_n = \{(1,\theta): -\pi \cdot 2^{1-2n} \leqslant \theta \leqslant -\pi \cdot 2^{-2n}\},$$

$$B_n = \{(1,\theta): \pi - \pi \cdot 2^{-2n} < \theta < \pi - \pi \cdot 2^{-2n-1}\},$$

$$C = \{(1,\theta): 0 \leqslant \theta < \pi/2\}.$$

Now, let

$$X = \left[\bigcup_{n=1}^{\infty} A_n\right] \cup \left[\bigcup_{n=1}^{\infty} B_n\right] \cup C.$$

The fact that $(X,\rho) \in \mathrm{UEP}$ can easily be seen from the following simple observation. If $(1,\theta_1)$ and $(1,\theta_2)$ are any two points on the unit circle, then the two points equidistant from them on the circle are diametrically opposite to one another; in fact, they are $(1,(\theta_1+\theta_2)/2)$ and $(1,(\theta_1+\theta_2)/2+\pi)$. It follows that (X,ρ) is a locally compact separable metric space having UEP but not satisfying (P.1) (the component C is not an open subset of X).

Presentation 3. Study Example 2 from Anthony D. Berard, Jr., Characterizations of metric spaces by the use of their midsets: Intervals, Fund. Math. 73 (1971 / 72), no. 1, 1—7. Why do you suppose that the Cantor ternary set is not used and set Y is created instead? Set Y is similar to set C in Ohta/Ono page 219.

EXAMPLE 2. A metric space X which is complete and possesses UMP but which is totally disconnected. Let Y be the set consisting of all points of $[0, 1]$, which when expressed to the base 4 possess no ones or twos in their expansion. Notice that Y is homeomorphic to the Cantor ternary set. Let h and k be linear homeomorphisms of Y into $[0, 1/4]\times\{0\}$ and $[1, 5/4]\times\{0\}$, $h\colon Y\to[0, 1/4]\times\{0\}$ and $k\colon Y\to[1, 5/4]\times\{0\}$. Let q be the point $(5/8, 1)$ and set $X=\{q\}\cup h(Y)\cup k(Y)$. Let $\sigma\colon X\times X\to R^+$ be defined by $\sigma(q, q)=0$, $\sigma(x, q)=\sigma(q, x)=1$ whenever $x\in h(Y)\cup k(Y)$, and $\sigma(x, y)=\varrho(x, y)$ where ϱ is the usual metric on R, whenever x and y belong to $h(Y)\cup k(Y)$. (X, σ) is a complete metric space with the unique midpoint property, but X is totally disconnected.

Presentation 4: In Ohta/Ono, verify statement (3.1), page 219.

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Everybody should look at this example. It need not be presented. It is from Nadler's paper (cited above). Note that UEP (unique equidistant property) is Nadler's notation for UMP.

(3.4) EXAMPLE. Let (X,σ) be any uncountable metric space having UEP. Change the metric σ to the metric d defined by the following formulas:

(1) $$d(x,y) = 1 + \sigma(x,y) \quad \text{for } x, y \in X \text{ with } x \neq y;$$

(2) $$d(x,y) = 0 \quad \text{if and only if } x = y.$$

It is easy to verify that (X,d) is a metric space and that (X,d) is discrete (i.e., $\{x\}$ is an open subset of X for each $x \in X$). It is also easy to see that, for any $x, y, z \in X$, $d(x,z) = d(y,z)$ is equivalent to $\sigma(x,z) = \sigma(y,z)$. From this equivalence and the fact that $(X,\sigma) \in$ UEP, it follows easily that $(X,d) \in$ UEP. Since (X,d) is uncountable and discrete, (X,d) is not separable and (hence) is not embeddable in R^1.