Immerse Topology Homework

(Exercises 1-22)

Exercise 1. Prove DeMorgan's Laws:

a. Show $\left(\bigcup_{\alpha} A_\alpha\right)^c = \bigcap_{\alpha} A_\alpha^c$

Proof. Now, $a \in \left(\bigcup_{\alpha} A_\alpha\right)^c$, if and only if $a \notin \bigcup_{\alpha} A_\alpha$, if and only if $a \notin A_\alpha\ \forall\alpha$, if and only if $a \in A_\alpha^c\ \forall\alpha$, if and only if $a \in \bigcap_{\alpha} A_\alpha^c$. □

b. Show $\left(\bigcap_{\alpha} A_\alpha\right)^c = \bigcup_{\alpha} A_\alpha^c$.

Proof. $a \in \left(\bigcap_{\alpha} A_\alpha\right)^c$ if and only if $a \notin \bigcap_{\alpha} A_\alpha$, if and only if $a \notin A_\alpha$ for some α, if and only if $a \in A_\alpha,$ for some α, if and only if $a \in \bigcup_{\alpha} A_\alpha^c$. □

Exercise 2. A. Let τ_c be the collection of all subsets U of X such that $X - U$ is countable or all of X. Show that τ_c is a topology on X.

Proof. $X - \emptyset = X$, and $X - X = \emptyset$, which is countable. Thus, $\emptyset \in \tau_c$ and $X \in \tau_c$.

Suppose $\{U_\alpha\}_{\alpha\in J}$ is a family of open sets in τ_c. If $U_\alpha = \emptyset$ for all $\alpha \in J$ then $\bigcup_{\alpha\in J} U_\alpha = \emptyset$. So $\bigcup_{\alpha\in J} U_\alpha$ is in τ_c. If there is some nonempty set in $\{U_\alpha\}_{\alpha\in J}$, then $X - \bigcup_{\alpha\in J} U_\alpha = \bigcap_{\alpha\in J}(X - U_\alpha)$, which is at most countably infinite because at least one of the sets in $X - U_\alpha$ is countable. So $\bigcup_{\alpha\in J} U_\alpha$ is in τ_c.

Now suppose we have $\{U_1, \ldots U_n\}$, a finite collection of open sets in τ_c. If $U_i = \emptyset$ for some $i \leqslant n$ then $\bigcap_{i=1}^{n} U_i = \emptyset$ which is in τ_c. If $U_i \neq \emptyset$ for all $i \leqslant n$, then $X - \bigcap_{i=1}^{n} U_i = \bigcup_{i=1}^{n}(X - U_i)$. This set is countable because it is the finite union of countable sets. Therefore $\bigcap_{i=1}^{n} U_i$ is in τ_c. □

B. Is the collection of all subsets $U \subseteq X$ such that $X - U$ is infinite, empty, or all of X a topology? No.

Proof. Consider $\mathbb{R}$ with this topology. Both $(-\infty, 2)$ and $(2, \infty)$ are open sets, as their complements are infinite. However, $(-\infty, 2) \cup (2, \infty) = \mathbb{R} - \{2\}$ is not an open set because its complement is finite.

$\square$

Exercise 3. Suppose $\{\tau_i\}_{i\in I}$ is a collection of topologies on X.

a. Show that $\bigcap_{i\in I} \tau_i$ is a topology.

We know X and $\emptyset$ are in τ_i $\forall i \in I$. So $X, \emptyset \in \bigcap_{i\in I} \tau_i$. Now let $\{U_\alpha\}_{\alpha\in A}$ be a collection of sets in $\bigcap_{i\in I} \tau_i$. Then $\forall \alpha \in A$, we know U_α is in τ_i $\forall i \in I$. So $\bigcup_{\alpha\in A} U_\alpha \in \tau_i$ $\forall i \in I$ since each τ_i is a topology. Hence $\bigcup_{\alpha\in A} U_\alpha \in \bigcap_{i\in I} \tau_i$. Finally, let $U, V \in \bigcap_{i\in I} \tau_i$. Then $U, V \in \tau_i$ $\forall i \in I$. So $U \cap V \in \tau_i$ $\forall i \in I$ since each τ_i is a topology. So $U \cap V \in \bigcap_{i\in I} \tau_i$. So finite intersections of elements of $\bigcap_{i\in I} \tau_i$ are in $\bigcap_{i\in I} \tau_i$. Hence $\bigcap_{i\in I} \tau_i$ is a topology. $\square$

b. Find a counterexample to show that the union of a collection of topologies on X need not a topology.

Let $X = \{a, b, c\}$, $\tau_1 = \{X, \emptyset, \{a\}, \{b, c\}\}$ and $\tau_2 = \{X, \emptyset, \{b\}, \{a, c\}\}$.

Then $\bigcup_{i=1}^{2} \tau_i = \{X, \emptyset, \{a\}, \{b\}, \{b, c\}, \{a, c\}\}$

But $\{b, c\} \cap \{a, c\} = \{c\}$ and $\{c\} \notin \bigcup_{i=1}^{2} \tau_i$. So $\bigcup_{i=1}^{2} \tau_i$ is not a topology.

c. Suppose that τ_i are a collection of topologies on a space X. Show there is a unique smallest topology containing all τ_i.

Proof. Consider the set $\mathcal{B} := \{\tau : \tau_i \subseteq \tau \ \forall i\}$. This set is non-empty as the discrete topology contains all the τ_i. Let

$$\mathcal{S} = \bigcap_{\tau\in\mathcal{B}} \tau.$$

This is a topology as you have already shown in previous work. This topology is contained in all topologies containing all the τ_i. Let τ be a topology containing all the τ_i, then $\tau \in \mathcal{B}$. Therefore,

$$\mathcal{S} \subseteq \tau.$$

To show that this topology is unique, suppose that there is another topology $\mathcal{T}$ with the property that if $\tau_i \subseteq \tau$ for all i, then $\mathcal{T} \subseteq \tau$. Now, $\tau_i \subseteq \mathcal{S}$ for all i, and thus $\mathcal{T} \subseteq \mathcal{S}$. But, $\tau_i \subseteq \mathcal{T}$, and so $\mathcal{T} \in \mathcal{B}$. Therefore, $\mathcal{S} \subseteq \mathcal{T}$.

So, we have that $\mathcal{S}$ is the unique smallest topology containing all the τ_i, as desired. □

Exercise 4. Let X={a,b,c}, τ={X,∅,{a},{a,b}} and $\tau\prime$={X,∅,{a},{b,c}}.

A) Find the smallest topology containing τ and τ'.

Solution: We know that both X, and $\emptyset$ must be in our topology. Since the union of the elements of τ and τ' must also be in our topology, we know that $\{a\}, \{a, b\}$ and $\{b, c\}$ must be in our topology. Since the intersection of the elements of τ and τ' must be in our topology, we know that $\{b\}$ must be in our topology as well. Therefore the smallest topology containing τ and τ' is $\{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$.

B) Find the largest topology contained in τ and τ'.

Solution: As in shown in problem 3c., the largest topology contained in τ and τ' is $\tau \cap \tau'$. Thus, the largest topology contained in τ and τ' is $\{X, \emptyset, \{a\}\}$

Exercise 5. Let X be a topological space. Show the following conditions hold:

a. The empty set and X are closed.

Proof. $X = \emptyset^c$ is closed since $\emptyset$ is open and $\emptyset = X^c$ is closed since X is open. □

b. Arbitrary intersections of closed sets are closed.

Proof. Let $\cap_{\alpha \in \Delta} A_\alpha$ be an arbitrary intersection of closed sets. Then $(\cap_{\alpha \in \Delta} A_\alpha)^c = \cup_{\alpha \in \Delta} A_\alpha^c$ is an arbitrary union of open sets and hence open. $\therefore \cap_{\alpha \in \Delta} A_\alpha$ is closed. □

c. Finite unions of closed sets are closed.

Proof. Let $\cup_{i=1}^n A_i$ be a finite union of closed sets. Then $(\cup_{i=1}^n A_i)^c = \cap_{i=1}^n A_i^c$ is a finite intersection of open sets and hence open. $\therefore \cup_{i=1}^n A_i$ is closed. □

Exercise 6. Show the following:

a. $\overline{A \cup B} = \overline{A} \cup \overline{B}$

Proof. Now, $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$. Also, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. Therefore, we have that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Now, let $x \in \overline{A} \cup \overline{B}$, then $x \in \overline{A}$ or $x \in \overline{B}$. Suppose that $x \in \overline{A}$; then for every open set $\mathcal{O}$ containing x, we have that $\mathcal{O} \cap A \neq \emptyset$. It follows that $\mathcal{O} \cap (A \cup B) \neq \emptyset$. Hence, we have that for every open set $\mathcal{O}$ containing x, then $\mathcal{O} \cap (A \cup B) \neq \emptyset$. Therefore $x \in \overline{A \cup B}$. Similarly, if $x \in \overline{B}$, then $x \in \overline{A \cup B}$

Therefore, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. □

b. $\overline{\bigcup_\alpha A_\alpha} \supset \bigcup \overline{A_\alpha}$

Proof. Suppose that $x \in \bigcup_\alpha \overline{A_\alpha}$. Then there is some α with $x \in \overline{A_\alpha}$. Thus every open set $\mathcal{O}$ which contains x has $\mathcal{O} \cap A_\alpha \neq \emptyset$. It follows that $\mathcal{O} \cap \bigcup_\alpha A_\alpha \neq \emptyset$. Thus, $x \in \overline{\bigcup_\alpha A_\alpha}$.

(Give an example where equality fails.)

Let $A_\alpha = (\frac{1}{\alpha}, \infty)$. Thus, $\overline{A_\alpha} = [\frac{1}{\alpha}, \infty)$. Now, $0 \notin \bigcup_\alpha \overline{A_\alpha}$, but $0 \in \overline{\bigcup_\alpha A_\alpha}$ □

Exercise 7. Let τ and τ' be two topologies on a set X and let $i : (X, \tau') \to (X, \tau)$ be the identity map.

A) τ' is finer than $\tau \Leftrightarrow i$ is continuous.

B)$\tau' = \tau \Leftrightarrow i$ is a homeomorphism.

Proof. A) First assume τ' is finer than τ. Let S be a set open in (X, τ). Then S is open in (X, τ'). Since $i : (X, \tau') \to (X, \tau)$ is the identity map, we have that $i^{-1}(S) = S$. Then $i^{-1}(S)$ is open in (X, τ'). Therefore i is continuous.

Next, let i be continuous. Let O be an open set in τ. Then since i is continuous, $i^{-1}(O)$ is open in (X, τ'). Since $i^{-1}(O) = O$, τ' is finer than τ. □

Proof. B) Assume $\tau = \tau'$. Then τ' is finer than τ. By part A, i is continuous. Since τ is also finer than τ', i^{-1} is also continuous. It is clear that i is both one-to-one and onto. Then, i is a homeomorphism.

Now, assume that i is a homeomorphism. Then i is continuous, so by part A, τ' is finer than τ. i^{-1} is also continuous, so τ is finer than τ'. Then, we must have $\tau = \tau'$. □

Exercise 8. *Let τ_n be the topology on the real line generated by the usual bases plus $\{n\}$. Show that (R, τ_1) and (R, τ_2) are homeomorphic, but that τ_1 does not equal τ_2.*

PROOF. Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x + 1$. To show f is injective let $x, y \in \mathbf{R}$ and assume that $f(x) = f(y)$. Then $f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$. To show that f is surjective let $c \in \mathbf{R}$. Then choose $a = c - 1 \in \mathbf{R}$. Therefore $f(a) = f(c-1) = c$.

To prove that f is a homeomorphism we have to show that both f and f^{-1} are continuous. Let U be a basic open set in τ_2. Then either there exist $a, b \in \mathbf{R}$ such that $U = (a, b)$, or $U = \{2\}$.

Case 1. $U = (a, b)$. Then $f^{-1}(U) = f^{-1}((a, b)) = (a-1, b-1)$. Notice that $(a-1, b-1)$ is an open set in τ_1.

Case 2. $U = \{2\}$. Then $f^{-1}(U) = f^{-1}(\{2\}) = \{1\}$. Notice $\{1\}$ is an open set in τ_1. Thus f is continuous function from $\mathbf{R}$ to $\mathbf{R}$.

Similarly, let O be a basic open set in τ_1. Then either there exist $c, d \in \mathbf{R}$ such that $O = (c, d)$, or $O = \{1\}$

Case 1.$O = (c, d)$. Then $f(O) = f((c, d)) = (c + 1, d + 1)$. Notice that $(c + 1, d + 1)$ is an open set in τ_2.

Case 2. $O = \{1\}$. Then $f(O) = f(\{1\}) = \{2\}$. Notice $\{2\}$ is an open set in τ_2.
Therefore f^{-1} is continuous from $\mathbf{R}$ to $\mathbf{R}$ Hence f is a homeomorphism.

The open set $\{1\}$ is open in τ_1, but not open in τ_2; therefore, $\tau_1 \neq \tau_2$.

Exercise 9. Build a function that is continuous at a single point.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$$

Let V be a neighborhood of $f(0) = 0$ in $\mathbb{R}$. Write $V = \cup_\alpha (a_\alpha, b_\alpha)$, as a union of basis elements. Some $(a_{\alpha_0}, b_{\alpha_0})$ contains 0. Choose $\hat{C} = min\{|a_{\alpha_0}|, b_{\alpha_0}\}$, which means $0 \in (-\hat{C}, \hat{C}) \subseteq (a_{\alpha_0}, b_{\alpha_0})$. It follows that $f((-\hat{C}, \hat{C})) \subseteq (-\hat{C}, \hat{C}) \subseteq (a_{\alpha_0}, b_{\alpha_0}) \subseteq \cup_\alpha (a_\alpha, b_\alpha)$. Since $(-\hat{C}, \hat{C})$ is a neighborhood of 0, by definition, we have that f is continuous at the point 0.

We now show that f is not continuous at any non-zero point. Suppose $x_0 \in \mathbb{R}$ is a non-zero point. WLOG, assume $x_0 > 0$. Then depending on whether x_0 is rational or irrational, its image is either positive or negative. So we can choose a neighborhood of its image, $f(x_0)$, that does not contain the point 0. Call this neighborhood W. What is important is that if we let U be an arbitrary neighborhood of x_0, we can say U has an open interval about x_0 inside of it, say (a, b). So, we can choose a positive irrational number between x_0 and b which lies in (a, b), say y. Note $f(y) < 0$. Also, we can pick some rational number inside of (a, b) between x_0 and b (which is positive) and will have a positive image. Both of these points will not lie in W. Hence f is not continuous at x_0.

Exercise 10. Show that if $X \subseteq Y \subseteq Z$ then the subspace topology on X as a subspace on Y is the same as the subspace topology on X as a subspace of Z

Let $(Y, \mathcal{T}_Y)$ and $(Z, \mathcal{T}_Z)$ be topological spaces and let $X \subseteq Y \subseteq Z$. We want to show that $(X, \mathcal{T}'_Y) = (X, \mathcal{T}'_Z)$ where $\mathcal{T}'_Y$ is the subspace topology of $\mathcal{T}_Y$ on X and $\mathcal{T}'_Z$ is the subspace topology of $\mathcal{T}_Z$ on X. Let $U \in \mathcal{T}'_Y$. So $U = U_Y \cap X$ where $U_Y \in \mathcal{T}_Y$. Since U_Y is open in Y, $U_Y = U_Z \cap Y$, where $U_Z \in \mathcal{T}_Z$ So $U = U_Y \cap X = U_Z \cap Y \cap X$. But $Y \cap X = X$, so $U = U_Z \cap X$ and hence is in $\mathcal{T}'_Z$. Thus $\mathcal{T}'_Y \subseteq \mathcal{T}'_Z$. Now let $V \in \mathcal{T}'_Z$. So $V = V_Z \cap X$ where $V_Z \in \mathcal{T}_Z$. We know $V_Z \cap Y = V_Y \in \mathcal{T}'_Y$. $V_Y \cap X = V_Z \cap Y \cap X$. But $Y \cap X = X$, so $V_Y \cap X = V_Z \cap X = V$. Since $V_Y \cap X \in \mathcal{T}'_Y$ we now know $V \in \mathcal{T}'_Y$. Thus $\mathcal{T}'_Z \subseteq \mathcal{T}'_Y$ so $\mathcal{T}'_Z = \mathcal{T}'_Y$. □

Exercise 11. Let$\mathcal{B} = \{(a, b) \times (c, d) : a < b \text{ and } c < d \text{ and } a, b, c, d \text{ are rationals}\}$. Show that $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

(1) Let $(x, y) \in \mathbb{R}^2$. Because the rationals are dense, there exists an $\epsilon_1, \epsilon'_1, \epsilon_2$, and ϵ'_2 such that $x - \epsilon_1 \in \mathbb{Q}$, $x + \epsilon'_1 \in \mathbb{Q}$, $y - \epsilon_2 \in \mathbb{Q}$, and $y + \epsilon'_2 \in \mathbb{Q}$. Let $a_x = x - \epsilon_1$, $b_x = x + \epsilon'_1$,

$c_x = y - \epsilon_2$, and $d_x = y + \epsilon_2'$ and $B_x = \{(a_x, b_x) \times (c_x, d_x)\}$. Then, $(x, y) \in B_x \subseteq \mathcal{B}$. Because of the density of $\mathbb{Q}$, there will always be such a B_x for any (x, y) in $\mathbb{R}$.

(2) Let $(x, y) \in B_1 \bigcap B_2$ where $(x, y) \in \mathbb{R}$ and $B_1, B_2 \in \mathcal{B}$ such that $B_1 = \{(a', b') \times (c', d')\}$ and $B_2 = \{(a'', b'') \times (c'', d'')\}$. Set $a = max(a', a'')$, $b = min(b', b'')$, $c = max(c', c'')$, and $d = min(d', d'')$. We can show that $a < b$ and $c < d$. First, $a' < b'$ and $a'' < b''$ is always true, so if $a = a'$ and $b = b'$ or $a = a''$ and $b = b''$, then $a < b$ follows. Since $B_1 \bigcap B_2 \neq \emptyset$, then $a'' < b'$ and likewise $a' < b''$. Therefore, if $a = a''$ and $b = b'$ or $a = a'$ and $b = b''$, it is still true that $a < b$. The same logic can be used to show that $c < d$. Then $B_3 = \{(a, b) \times (c, d)\}$ is an element of $\mathcal{B}$. Due to the definition of B_3, $(x, y) \in B_3 \subseteq B_1 \bigcap B_2$.

Since $\mathcal{B}$ meets the two requirements of the definition of basis, $\mathcal{B}$ is a basis for $\mathbb{R}^2$. $\square$

Exercise 12. Determine which of the following equations hold. If not, determine whether any inclusion holds.
A) $\overline{A \cap B} = \overline{A} \cap \overline{B}$
B) $\overline{\bigcap A_\alpha} = \bigcap \overline{A_\alpha}$
C) $\overline{A - B} = \overline{A} - \overline{B}$
D) $(A \cup B)' = A' \cup B'$
E) $(A \cap B)' = A' \cap B'$

Solutions:

A) Claim: $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ only.
Proof: Let $x \in \overline{A \cap B} = (A \cap B) \cup (A \cap B)'$
If $x \in (A \cap B)$ then $x \in A \subseteq \overline{A}$ and $x \in B \subseteq \overline{B}$, so $x \in \overline{A} \cap \overline{B}$
If $x \in (A \cap B)'$ then $\forall$ U open sets such that $x \in U$, $U \cap (A \cap B) \setminus \{x\} \neq \emptyset$
Now, $(A \cap B) \subseteq A \Rightarrow (A \cap B) \setminus \{x\} \subseteq A \setminus \{x\} \Rightarrow U \cap (A \cap B) \setminus \{x\} \subseteq U \cap A \setminus \{x\}$
$\Rightarrow U \cap A \setminus \{x\} \neq \emptyset \Rightarrow x \in A' \subseteq \overline{A}$
And, $(A \cap B) \subseteq B \Rightarrow (A \cap B) \setminus \{x\} \subseteq B \setminus \{x\} \Rightarrow U \cap (A \cap B) \setminus \{x\} \subseteq U \cap B \setminus \{x\}$
$\Rightarrow U \cap B \setminus \{x\} \neq \emptyset \Rightarrow x \in B' \subseteq \overline{B}$ so $x \in \overline{A} \cap \overline{B}$
In either case then, $x \in \overline{A} \cap \overline{B}$
$\therefore \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ $\square$

Example where $\overline{A \cap B} \not\supseteq \overline{A} \cap \overline{B}$
Let $A = (0, 1)$ and $B = (1, 2)$ then $A \cap B = \emptyset \Rightarrow \overline{A \cap B} = \emptyset$
But $\overline{A} = [0, 1]$ and $\overline{B} = [1, 2]$ so $\overline{A} \cap \overline{B} = \{1\}$
Since $\emptyset \not\supseteq \{1\}$, $\overline{A \cap B} \not\supseteq \overline{A} \cap \overline{B}$

B) Claim: $\overline{\bigcap A_\alpha} \neq \bigcap \overline{A_\alpha}$.
(Note: $\alpha \in \mathbb{N}$)
Suppose that $x \in \overline{\bigcap A_\alpha}$. Then either $x \in \bigcap A_\alpha$ or x is a limit point of $\bigcap A_\alpha$. If $x \in \bigcap A_\alpha$, then $x \in A_\alpha$ for every α and, therefore $x \in \overline{A_\alpha}$ for every $oalpha$. Thus $x \in \bigcap \overline{A_\alpha}$. On the ether hand, if x is a limit point of $\bigcap A_\alpha$, then there is a sequence $\{x_n\}$ in $\bigcap A_\alpha$ which converges to x. But then $\{x_n\} \subseteq A_\alpha$ for each α. Thus x is a limit point of A_α for each α. It

follows that $x \in \bigcap \overline{A_\alpha}$.

Example where $\overline{\bigcap A_\alpha} \not\supseteq \bigcap \overline{A_\alpha}$

Let $A_\alpha = (\alpha, \alpha+1)$

Then $\bigcap A_\alpha = \emptyset$ so $\overline{\bigcap A_\alpha} = \emptyset$

But $\overline{A_\alpha} = [\alpha, \alpha+1]$ so $\bigcap \overline{A_\alpha} = \{1, 2, 3, ...\}$

Since $\emptyset \not\supseteq \{1, 2, 3, ...\}$, $\overline{\bigcap A_\alpha} \not\supseteq \bigcap \overline{A_\alpha}$

C) Claim: $\overline{A - B} \supseteq \overline{A} - \overline{B}$ only.

Proof: Let $x \in \overline{A} - \overline{B}$ So $x \in \overline{A}$ but $x \notin \overline{B}$

So ($x \in A$ or $x \in A'$) and ($x \notin B$ and $x \notin B'$)

If $x \in A$ then since $x \notin B$, $x \in A - B$ which implies $x \in \overline{A - B}$.

If, on the other hand, $x \in A'$, then every neighborhood of x intersected with $A \setminus \{x\}$ is nonempty.

And $\exists V$ neighborhood of x such that $V \cap B = \emptyset$

Let U be any neighborhood of x.

Assume $U \cap (A - B) = \emptyset$; then, since $U \cap A \neq \emptyset$, $U \cap (A \cap B) \neq \emptyset$.

But $U \cap V$ is a neighborhood of x such that $(V \cap U) \cap A = \emptyset \rightarrow\leftarrow$

So $U \cap (A - B) \neq \emptyset \Rightarrow x \in (A - B)' \subseteq \overline{A - B}$

$\therefore \overline{A - B} \supseteq \overline{A} - \overline{B}$ $\square$

Example where $\overline{A - B} \not\subseteq \overline{A} - \overline{B}$

Let $A = (0, 2)$ and $B = (1, 3)$. Then $A - B = (0, 1]$ and $\overline{A - B} = [0, 1]$

Now $\overline{A} = [0, 2]$ and $\overline{B} = [1, 3]$ So $\overline{A} - \overline{B} = [0, 1)$

Since $[0, 1] \not\subseteq [0, 1)$, $\overline{A - B} \not\subseteq \overline{A} - \overline{B}$

D) Claim: $(A \cup B)' = A' \cup B'$

Proof: First let $x \in (A \cup B)'$. Then there is a sequence $\{x_n\}$ in $A \cup B$ which converges to x. At least one of the sets A or B (WLOG assume that it is A) must contain infinitely many terms of the sequence $\{x_n\}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ with all its terms in A. Since $\{x_{n_i}\}$ converges to x, $x \in A'$. Thus, $(A \cup B)' \subseteq A' \cup B'$.

Now let $x \in A' \cup B'$ then $x \in A'$ or $x \in B'$.

If $x \in A'$ then $\forall$ U open sets such that $x \in U$, $U \cap A \setminus \{x\} \neq \emptyset$

Now $A \subseteq A \cup B \Rightarrow A \setminus \{x\} \subseteq A \cup B \setminus \{x\} \Rightarrow U \cap A \setminus \{x\} \subseteq U \cap (A \cup B) \setminus \{x\}$

$\Rightarrow U \cap (A \cup B) \setminus \{x\} \neq \emptyset$ So $x \in (A \cup B)'$

If $x \in B'$ then $\forall$ U open sets such that $x \in U$, $U \cap B \setminus \{x\} \neq \emptyset$

Now $B \subseteq A \cup B \Rightarrow B \setminus \{x\} \subseteq A \cup B \setminus \{x\} \Rightarrow U \cap B \setminus \{x\} \subseteq U \cap (A \cup B) \setminus \{x\}$

$\Rightarrow U \cap (A \cup B) \setminus \{x\} \neq \emptyset$ So $x \in (A \cup B)'$

In either case $x \in (A \cup B)'$ so $(A \cup B)' \supseteq A' \cup B'$.

$\therefore (A \cup B)' = A' \cup B'$ $\square$

E) Claim: $(A \cap B)' \subseteq A' \cap B'$ only.

Proof: Let $x \in (A \cap B)'$ then $\forall$ U open sets such that $x \in U$, $U \cap (A \cap B) \setminus \{x\} \neq \emptyset$

Now, $(A \cap B) \subseteq A \Rightarrow (A \cap B) \setminus \{x\} \subseteq A \setminus \{x\} \Rightarrow U \cap (A \cap B) \setminus \{x\} \subseteq U \cap A \setminus \{x\}$
$\Rightarrow U \cap A \setminus \{x\} \neq \emptyset \Rightarrow x \in A'$
And, $(A \cap B) \subseteq B \Rightarrow (A \cap B) \setminus \{x\} \subseteq B \setminus \{x\} \Rightarrow U \cap (A \cap B) \setminus \{x\} \subseteq U \cap B \setminus \{x\}$
$\Rightarrow U \cap B \setminus \{x\} \neq \emptyset \Rightarrow x \in B'$
So $x \in A' \cap B'$
$\therefore (A \cap B)' \subseteq A' \cap B'$ $\square$

Example where $(A \cap B)' \not\supseteq A' \cap B'$
Let $A = (0, 1)$ and $B = (1, 2)$ then $A \cap B = \emptyset \Rightarrow (A \cap B)' = \emptyset$
But $A' = [0, 1]$ and $B' = [1, 2]$ so $A' \cap B' = \{1\}$
Since $\emptyset \not\supseteq \{1\}$, $(A \cap B)' \not\supseteq A' \cap B'$

Exercise 13. If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, what does the connectedness of X in one topology imply about the connectedness of X in the other?

If X is connected in $\mathcal{T}_1$, this implies that X is connected in $\mathcal{T}_2$. For if X is not connected in $\mathcal{T}_2$, there exist nonempty open sets U and V in $\mathcal{T}_2$ such that $X \subseteq (U \cup V)$ and $U \cap V = \emptyset$. But then $U, V \in \mathcal{T}_1$ so X is not connected in $\mathcal{T}_1$ either.

But inclusion does not hold the other way. For example, in the trivial topology on $\mathbb{R}$, every set is connected, but not every set is connected in the finer usual metric.

Exercise 14. Let A_n be a sequence of connected sets such that A_n intersects A_{n+1} nontrivially for each n. Show that $\bigcup_{n=1}^{N} A_n$ is connected.

Proof. We proceed by induction on N. Base Case: $N = 2$

$A_1 \cap A_2 \neq \emptyset$, so $A_1 \cup A_2$ is a union of connected sets with at least one point in common. By "connectedness theorem (d)", it follows that $\bigcup_{n=1}^{2} A_n$ is connected.

Inductive Step: Assume that $\bigcup_{n=1}^{N-1} A_n$ is connected. We now want to show that $\bigcup_{n=1}^{N} A_n$ is connected.

$A_{N-1} \subseteq \bigcup_{n=1}^{N-1} A_n$ and $A_{N-1} \cap A_N \neq \emptyset$ by our assumption, therefore $(\bigcup_{n=1}^{N-1} A_n) \cap A_N \neq \emptyset$.

This implies that $\bigcup_{n=1}^{N-1} A_n \cup A_N$ is connected. $\bigcup_{n=1}^{N-1} A_n \cup A_N = \bigcup_{n=1}^{N} A_n$, and we conclude that $\bigcup_{n=1}^{N} A_n$ is connected for every N.

Next, we would like to show that $\bigcup_{n=1}^{\infty} A_n$ is connected. We argue by contradiction. Assume that $\bigcup_{n=1}^{\infty} A_n$ is not connected. Then there exist two nonempty sets, call them B_1 and B_2, such that $B_1 \cap B_2 = \emptyset$ and $B_1 \cup B_2 = \bigcup_{n=1}^{\infty} A_n$ (B_1 and B_2 form a separation on $\bigcup_{n=1}^{\infty} A_n$). Let us consider the location of A_1. Since A_i is connected for each $i = \{1, ...n\}$, is must be the case that A_i is completely in one of B_1 or B_2. Without loss of generality, let $A_1 \subseteq B_1$. Given that $B_1 \cup B_2$ is a non-trivial separation of $\bigcup_{n=1}^{\infty} A_n$, there exists $A_j \subseteq B_2$ for some collection of $j \in \{1, 2, 3, ...\}$ (it is possible that only a single A_j be in B_2). Choose the smallest of these j where $A_j \subseteq B_2$. Call this $A_{j'}$. Therefore we have $B_1 \supseteq \bigcup_{n=1}^{j'-1} A_n$ and $B_2 \supseteq A_j$. Then $A_{j'-1} \subseteq B_1$, and since $B_1 \cap B_2 = \emptyset$, $A_{j'-1} \cap A_{j'} = \emptyset$. This is a contradiction, since we assumed that A_n intersects A_{n+1} nontrivially for each n. We conclude that $\bigcup_{n=1}^{\infty} A_n$ is connected.

□

Exercise 15. *If X is an infinite set then it is connected in the finite complement topology.*

Proof. (by contrapositive) Let X be a disconnected set in the finite complement topology, $\tau = \{U \subseteq X \mid X - U$ is either finite or $X\}$. Since X is disconnected, there exist $U_1 \in \tau$ and $U_2 \in \tau$ such that $U_1 \neq \emptyset$, $U_2 \neq \emptyset$, $U_1 \cup U_2 = X$, and $U_1 \cap U_2 = \emptyset$. Since $U_1 \cup U_2 = X$ and $U_1 \cap U_2 = \emptyset$, $X - U_1 = U_2$. We know that $U_1 \neq \emptyset$, so $U_2 = X - U_1 \neq X$. Since $U_1 \in \tau$ and $X - U_1 \neq X$, $X - U_1$ is finite. Therefore, U_2 is finite. Similarly, U_1 is also finite. Since $U_1 \cup U_2 = X$ and the union of finite sets is finite, X must be finite. □

Exercise 16. If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, what does the compactness of X in one topology imply about the compactness of X in the other?

Let $\mathcal{T}_1$ be finer than $\mathcal{T}_2$. So $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

Claim 1: If $(X, \mathcal{T}_1)$ compact, then $(X, \mathcal{T}_2)$ compact.

Let $(X, \mathcal{T}_1)$ be compact and let $\{U_\alpha\}$ be an open cover of $(X, \mathcal{T}_2)$. Then $\{U_\alpha\}$ is an open cover of $(X, \mathcal{T}_1)$. So $\exists$ $U_1, ..., U_n$, a finite open subcover of $(X, \mathcal{T}_1)$. Since $U_1, ..., U_n \in \{U_\alpha\}$, we know $U_1, ..., U_n \in \mathcal{T}_2$. So $U_1, ..., U_n$ are a finite open subcover of $(X, \mathcal{T}_2)$. Thus $(X, \mathcal{T}_1)$ compact implies $(X, \mathcal{T}_2)$ compact. □

Claim 2: $(X, \mathcal{T}_2)$ compact does not necessarily mean $(X, \mathcal{T}_1)$ compact.

Consider $X = [0, 1], \mathcal{T}_2 =$ the usual topology on $\mathbb{R}$, and $\mathcal{T}_1 = \mathcal{P}(X)$. $\mathcal{T}_2 \subseteq \mathcal{T}_1$ and $(X, \mathcal{T}_2)$ is compact. But $\{x\} \in \mathcal{T}_1$ for all $x \in X$, so $S = \{\{x\} | 0 \leq x \leq 1\}$ is an open cover of $(X, \mathcal{T}_1)$ with no finite subcover. Thus $(X, \mathcal{T}_1)$ is not compact.

Exercise 17. Show the following:

(A) $Bd(A)$ is empty iff A is both open and closed.

($\Rightarrow$) Let $Bd(A)$ be empty; then $Bd(A) = \overline{A} \cap \overline{A^c} = \emptyset$. Then we can replace the closure of A and A^c with the unions of each set and its limit points giving us the following statement.

$$(i)\ Bd(A) = (A \cup A') \cap (A^c \cup (A^c)') = \emptyset$$

From (i) we can conclude that $A \cap (A^c)' = \emptyset$, which means that the set $(A^c)'$ is contained in A^c. Therefore A^c is closed and A is open. Also from (i) we can conclude that $A^c \cap A' = \emptyset$, which means that A' is contained in A. It follows that A is closed and A^c is open. Thus A is both open and closed.

($\Leftarrow$) Let A be open and closed; Then $A = \overline{A}$ and $A^c = \overline{(A^c)}$. This means that the boundary of A is empty, since $Bd(A) = \overline{A} \cap \overline{A^c} = A \cap A^c = \emptyset$. Hence, A being both open and closed implies that the $Bd(A) = \emptyset$.

(B) A is open iff Bd(A)= $\overline{A} - A$.

($\Rightarrow$)Let A be open. Then A^c is closed, so $(A^c)'$ is contained in A^c and $A^c = \overline{A^c}$. Then $Bd(A) = \overline{A} \cap \overline{A^c} = \overline{A} \cap A^c = \overline{A} - A$.

($\Leftarrow$) Suppose A is not open. Then there exists an $x \in A$ such that x is not an interior point of A. Thus for every open set U containing x, $U \not\subseteq A$. Thus x is a limit point of A^c. It follows that $x \in \overline{A} \cap \overline{A^c} =$ Bd(A). However, $x \notin \overline{A} - A$, so $Bd(A) \neq \overline{A} - A$.

Exercise 18. For any subset of the real line (with the usual topology) there are at most 14 sets (including A) that can be formed by using complementation and closure. Prove this by completing the following steps:

A. Show that if A is open then $\bar{A} = A^{-c-c-}$.

Proof.
•<u>Claim:</u> For any open set A, $A \subset A^{-c-c}$.

Since A is open, A^c is closed. Since $A \subset \bar{A}$, $A^c \supset A^{-c}$.
A^{-c-} is the smallest closed set containing A^{-c} and A^c is a closed set containing A^{-c}, so $A^{-c-} \subset A^c$.
Therefore $A = (A^c)^c \subset (A^{-c-})^c = A^{-c-c}$.

$\Rightarrow$ Since we are given that A is open, by the above claim $A \subset A^{-c-c}$. Then $A^{-c-c-} \supset A^{-c-c} \supset A$, so A^{-c-c-} is an closed set containing A, which implies $\bar{A} \subset A^{-c-c-}$.
$\Leftarrow$ Proof by contrapositive. If $x \notin \bar{A}$, then $x \in A^{-c}$. By the above claim, since A^{-c} is the complement of a closed set and therefore open, $A^{-c} \subset (A^{-c})^{-c-c}$. So $x \in A^{-c-c-c}$, which implies $x \notin A^{-c-c-}$. □

B. Let

$$K = \{A, \bar{A}, A^{-c}, A^{-c-}, A^{-c-c}, A^{-c-c-}, A^{-c-c-c}, \\ A^c, A^{c-}, A^{c-c}, A^{c-c-}, A^{c-c-c}, A^{c-c-c-}, A^{c-c-c-c}\}.$$

Show that K is closed under complementation and closure.

Proof. We need to show that the complement and the closure of each element of K is some other element of K. We demonstrate this in the chart below, using the fact that $A^{cc} = A$, $A^{--} = \bar{A}$ since $\bar{A}$ is already closed, and by part (a) if A is open then $\bar{A} = A^{-c-c-}$.

Element	Closure	Complement
A	A^{-}	A^{c}
A^{-}	$A^{--} = A^{-}$	A^{-c}
A^{-c}	A^{-c-}	$A^{-cc} = A^{-}$
A^{-c-}	$A^{-c--} = A^{-c-}$	A^{-c-c}
A^{-c-c}	A^{-c-c-}	$A^{-c-cc} = A^{-c-}$
A^{-c-c-}	$A^{-c-c--} = A^{-c-c-}$	A^{-c-c-c}
A^{-c-c-c}	$A^{-c-c-c-} = A^{-c-}$	$A^{-c-c-cc} = A^{-c-c-}$
A^{c}	A^{c-}	$A^{cc} = A$
A^{c-}	$A^{c--} = A^{c-}$	A^{c-c}
A^{c-c}	A^{c-c-}	$A^{c-cc} = A^{c-}$
A^{c-c-}	$A^{c-c--} = A^{c-c-}$	A^{c-c-c}
A^{c-c-c}	A^{c-c-c-}	$A^{c-c-cc} = A^{c-c-}$
A^{c-c-c-}	$A^{c-c-c--} = A^{c-c-c-}$	$A^{c-c-c-c}$
$A^{c-c-c-c}$	$A^{c-c-c-c-} = A^{c-c-}$	$A^{c-c-c-cc} = A^{c-c-c-}$

Since K is closed under complementation and closure, we can get at most 14 sets from any set in $\mathbb{R}$ using these two operations. □

C. Show that there is a set $A \subset \mathbb{R}$ such that K has exactly 14 distinct elements.

Proof. Let $A = ([0,1] \cap \mathbb{Q}) \cup [2,3) \cup (3,4) \cup \{5\}$. Below I demonstrate the 14 distinct sets formed from A by complementation and closure.

$A = ([0,1] \cap \mathbb{Q}) \cup [2,3) \cup (3,4) \cup \{5\}$
$\bar{A} = [0,1] \cup [2,4] \cup \{5\}$
$A^{-c} = (-\infty,0) \cup (1,2) \cup (4,5) \cup (5,\infty)$
$A^{-c-} = (-\infty,0] \cup [1,2] \cup [4,\infty)$
$A^{-c-c} = (0,1) \cup (2,4)$
$A^{-c-c-} = [0,1] \cup [2,4]$
$A^{-c-c-c} = (-\infty,0) \cup (1,2) \cup (4,\infty)$
$A^{c} = (-\infty,0) \cup ([0,1] \cap (\mathbb{R} \backslash \mathbb{Q})) \cup (1,2) \cup \{3\} \cup (4,5) \cup (5,\infty)$
$A^{c-} = (-\infty,2] \cup \{3\} \cup [4,\infty)$
$A^{c-c} = (2,3) \cup (3,4)$
$A^{c-c-} = [2,4]$
$A^{c-c-c} = (-\infty,2) \cup (4,\infty)$
$A^{c-c-c-} = (-\infty,2] \cup [4,\infty)$
$A^{c-c-c-c} = (2,4)$

□

Exercise 19. For any subset of the real line (with the usual topology) there are at most 7 sets (including A) that can be formed by using the interior and closure operations. Prove this by completing the following steps:

A. Show that $A^{o-} = A^{o-o-}$ and $A^{-o} = A^{-o-o}$.

Proof. We prove both statements by double containment. To complete both proofs we repeatedly use the facts that the interior of a set A is the *largest* open set contained in A and the closure is the *smallest* closed set containing A. So if we know $A \subset B$, this implies that $A^o \subseteq B^o$ and $\bar{A} \subseteq \bar{B}$. Also, since A^o is open $A^{oo} = A^o$ and since $\bar{A}$ is closed $A^{--} = \bar{A}$.

First we show $A^{o-} = A^{o-o-}$.

$\subseteq$ By definition of closure $A^o \subseteq A^{o-}$. Taking the interior of both sets gives $A^{oo} = A^o \subseteq A^{o-o}$. Then taking closures of both sets gives $A^{o-} \subseteq A^{o-o-}$.

$\supseteq$ By definition of interior $A^{o-o} \subseteq A^{o-}$. Taking closures of both sides gives $A^{o-o-} \subseteq A^{o--} = A^{o-}$.

Second we prove $A^{-o} = A^{-o-o}$.

$\subseteq$ By definition of closure $A^{-o} \subset A^{-o-}$. Taking the interior of both sets gives $A^{-oo} = A^{-o} \subset A^{-o-o}$.

$\supseteq$ By definition of interior $A^{-o} \subset A^{-}$. Taking closures of both sides gives $A^{-o-} \subset A^{--} = \bar{A}$. Taking the interior of both sides we get $A^{-o-o} \subset A^{-o}$.

□

B. Let $K = \{A, \bar{A}, A^{-o}, A^{-o-}, A^o, A^{o-}, A^{o-o}\}$. Show that K is closed under the interior and closure operations.

Proof. We need to show that the interior and the closure of each element of K is some other element of K. We demonstrate this in the chart below, using the fact that $A^{oo} = A^o$ since A^o is already open, $A^{--} = \bar{A}$ since $\bar{A}$ is already closed, and the two results from part (a).

Element	Closure	Interior
A	A^-	A^o
A^-	$A^{--} = A^-$	A^{-o}
A^{-o}	A^{-o-}	$A^{-oo} = A^{-o}$
A^{-o-}	$A^{-o--} = A^{-o-}$	$A^{-o-o} = A^{-o}$
A^o	A^{o-}	$A^{oo} = A^o$
A^{o-}	$A^{o--} = A^{o-}$	A^{o-o}
A^{o-o}	$A^{o-o-} = A^{o-}$	$A^{o-oo} = A^{o-o}$

Since K is closed under the interior and closure operation, we can get at most 7 sets from any set in $\mathbb{R}$ using these two operations. □

C. Show that there is a set $A \subset \mathbb{R}$ such that K has exactly 7 distinct elements.

Proof. Let $A = ([0,1] \cap \mathbb{Q}) \cup [2,3) \cup (3,4) \cup \{5\}$. Below I demonstrate the 7 distinct sets formed from A by the interior and closure operations.

$A = ([0,1] \cap \mathbb{Q}) \cup [2,3) \cup (3,4) \cup \{5\}$

$\bar{A} = [0,1] \cup [2,4] \cup \{5\}$

$A^{-o} = (0,1) \cup (2,4)$
$A^{-o-} = [0,1] \cup [2,4]$
$A^{o} = (2,3) \cup (3,4)$
$A^{o-} = [2,4]$
$A^{o-o} = (2,4)$

□

Exercise 20. Show that the product of two Hausdorff spaces is Hausdorff.

Proof. Let X and Y be Hausdorff spaces, and consider two distinct points in $X \times Y$, which we'll call p and q. We know we can write $p = p_x \times p_y$ and $q = q_x \times q_y$ for $p_x, q_x \in X,\ p_y, q_y \in Y$. It cannot be the case that $p_x = q_x$ and $p_y = q_y$ because then $p = q$. Thus, suppose without loss of generality that $p_x \neq q_x$. Since X is Hausdorff, we know we can find open sets $P_x, Q_x \subset X$ such that $p_x \in P_x,\ q_x \in Q_x$, and $P_x \cap Q_x = \emptyset$. Additionally, we can find open sets $P_y, Q_y \subset Y$ such that $p_y \in P_y, q_y \in Q_y$. Notice that $P_y \cap Q_y$ may be non-empty. We can then let $P = P_x \times P_y,\ Q = Q_x \times Q_y$, so $p \in P,\ q \in Q$, and $P \cap Q = \emptyset$. Since we have separated two arbitrary points in $X \times Y$ with disjoint open sets, $X \times Y$ is Hausdorff, so the product of two Hausdorff spaces is Hausdorff. □

Exercise 21. Extreme Value Theorem *Let X be compact and Y be ordered with the order topology. Let $f : X \to Y$ be continuous. Show there exists $a, b, \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*

Let $Z = f(X)$. As Y is ordered, Z has a least upper bound in Y; call it M . Choose $c \in Z$ with $c < z$ for every $z \in Z$ (if no such c exists, let c be the least element of Y). Consider the collection $\{\mathcal{O}_y : y \in Z\}$, where $\mathcal{O}_y = (c, y)$ (or $[c, y)$, if c is the least element of Y). Notice that for any $z \in Z$ with $z < M$ there exists some $y_z \in Z$ with $z < y_z$, ; otherwise z would be an upper bound for Z. So $z \in (c, y_z)$. If $M \notin Z$, the collection $\{\mathcal{O}_y : y \in Z\}$ is an open cover for Z. Note that the union of any finite subcollection $(c, y_1), \ldots (c, y_n)$ is (c, y_{max}), where $y_{max} = \max\{y_1, \ldots, y_n\}$. But $y_{max} < M$, and so, as argued above, there is some $w \in Z, w > y_{max}$. Thus $(c, y_{max}) \neq Z$. That is, there is no finite subcollection of $\{\mathcal{O}_y : y \in Z\}$ which covers Z. Therefore Z is not compact. However, Z is the continuous image of a compact set, and is therefore compact. It follows that $M \in Z$, and so there is some $b \in X$ such that $f(b) = M$.

A similar argument shows that there is an $a \in X$ such that $f(a)$ is the least upper bound of Z in Y.

Exercise 22. Show that X is compact if and only if every collection of closed sets with the finite intersection property has non-empty intersection. A collection of sets has the finite intersection property if and only if every finite sub-collection of sets has non-empty intersection.

Proof: We'll prove the contrapositive. X is not compact $\iff$ there exists a collection of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ such that $\bigcup_{\alpha \in \Lambda} U_\alpha = X$ but $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} \neq X$ for any $\alpha_1, \ldots, \alpha_n \in \Lambda$ $\iff$ there exists a collection of open sets $\{U_\alpha\}_{\alpha \in \Lambda}$ such that $(\bigcup_{\alpha \in \Lambda} U_\alpha)^c = \emptyset$ but $(U_{\alpha_1} \cup \cdots \cup$

$U_{\alpha_1})^c \neq \emptyset$ for any $\alpha_1, \ldots, \alpha_n \in \Lambda \iff$ there exists a collection of closed sets $\{U_\alpha^c\}_{\alpha \in \Lambda}$ such that $(\bigcap_{\alpha \in \Lambda} U_\alpha^c) = \emptyset$ but $(U_{\alpha_1}^c \cap \cdots \cap U_{\alpha_1}^c) \neq \emptyset$ for any $\alpha_1, \ldots, \alpha_n \in \Lambda \iff$ there exists a collection of closed sets with the finite intersection property which has empty intersection.